Differential Equations

Solutions of First Order Differential Equations

1. Differential Equations DE's

A differential equation is an equation involving an unknown function and its derivatives.

A differential equation is an **ordinary differential equation ODE** if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a **partial differential equation PDE**.

Example 1: The following are differential equations involving the unknown function *y*.

$$\frac{dy}{dx} = 5x + 3$$

$$e^{y}\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1$$

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$$

ODE

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0$$

PDE

The order of a differential equation is the order of the highest derivative appearing in the equation.

Example 2: From example 1, Equation 1 is a first-order differential equation; 2, 4, and 5 are second-order differential equations. (Note in 1.4 that the order of the highest derivative appearing in the equation is two.) Equation 3 is a third-order differential equation.

$$\frac{dy}{dx} = 5x + 3$$ **1st order DE**

$$e^{y}\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1$$ **2nd order DE**

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0$$ **3rd order DE**

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$$ **2nd order DE**

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0$$ **2nd order DE**

1.1 Notation of Differential Equations DE's

i. The expressions $y', y'', y''', y^4, \ldots\ldots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, . . . nth derivatives of y with respect to the independent variable under consideration.

ii. Thus, y'' represents $\frac{d^2y}{dx^2}$ if the independent variable is x, but represents $\frac{d^2y}{dp^2}$ if the independent variable is p.

iii. Observe that parenthesis are used in $y^{(n)}$ to distinguish it from the nth power, y^n.

iv. If the independent variable is time, usually denoted by t, primes are often replaced by dots. Thus, $\dot{y}, \ddot{y}, and\ \dddot{y}$ represent, $\frac{dy}{dt}, \frac{d^2y}{dt^2}$ and $\frac{d^3y}{dt^3}$ respectively.

Example 3: Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$, where c_1 and c_2 are arbitrary constants, a solution of $y'' + 4y = 0$?

Solution:

$$y' = 2c_1 \cos 2x - 2c_2 \sin 2x$$

$$y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$$

Hence

$$y'' + 4y = (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x)$$

$$= -4c_1 \sin 2x - 4c_2 \cos 2x + 4c_1 \sin 2x + 4c_2 \cos 2x$$

$$= 0$$

Then $y(x) = c_1 \sin 2x + c_2 \cos 2x$ is a solution of the D.E.

Example 4: Show that $y = 3e^{2x} - e^{-2x}$ is a solution for the DE, $y'' - 4y = 0$
Solution:

$y' = 3 * 2 * e^{2x} - (-2)e^{-2x} = 6e^{2x} + 2e^{-2x}$

$y'' = 6 * 2 * e^{2x} + 2 * (-2)e^{-2x} = 12e^{2x} - 4e^{-2x}$

Hence

$y'' - 4y = (12e^{2x} - 4e^{-2x}) - 4(3e^{2x} - e^{-2x})$

$= 12e^{2x} - 4e^{-2x} - 12e^{2x} + 4e^{-2x}$

$= 0$

Then $y = 3e^{2x} - e^{-2x}$ is a solution of the D.E

Homework

Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$.

2. First Order Differential Equations

To solve the first order differential equations, we have the following cases:

1. Separable Equations
2. Homogeneous Equations
3. Exact Equations
4. Linear Equations
5. Bernoulli Equations

2.1 Separable Equations

In this case the D.E can be written in the form:

$$f(x)dx + g(y)dy = 0 \dots\dots\dots (1)$$

Or $\quad Mdx + Ndy = 0$, where M=f(x) and N=g(y)

Solution of this equation by direct integration of both sides gives the general solution.

$$\int \boldsymbol{f(x)dx} + \int \boldsymbol{g(y)dy} = \boldsymbol{c} \dots\dots\dots \ (\mathbf{2})$$

Where, c is a constant.

Example 5: Solve the D.E, $(x+1)\frac{dy}{dx} = y$?

Solution:

$\frac{dy}{dx} = \frac{y}{(x+1)}$ Multiply both side by $\frac{dx}{y}$ to separate the variable and integrate both sides gives,

$$\frac{dy}{dx} \times \frac{dx}{y} = \frac{y}{(x+1)} \times \frac{dx}{y}$$

$$\frac{dy}{y} = \frac{dx}{(x+1)}$$

$$\int \frac{dy}{y} = \int \frac{dx}{(x+1)}$$

$$\ln(y) = \ln(x+1) + c$$

Example 6: Solve the D.E, $x(2y-3)dx + (x^2+1)dy = 0$?

Solution:

Separate the variables by multiply both side by $\frac{1}{(2y-3)(x^2+1)}$

$$\frac{1}{(2y-3)(x^2+1)} \times x(2y-3)dx + \frac{1}{(2y-3)(x^2+1)} \times (x^2+1)dy = 0$$

$$\frac{x}{(x^2+1)}dx + \frac{1}{(2y-3)}dy = 0$$

Integrate both sides,

$$\int \frac{x}{(x^2+1)}dx + \int \frac{1}{(2y-3)}dy = 0$$

$$\frac{1}{2}\int \frac{2x}{(x^2+1)}dx + \frac{1}{2}\int \frac{2}{(2y-3)}dy = 0$$

$$\frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(2y-3) + c = 0$$

Homework

Solve the D.E, $xe^x dy + \frac{x^2+1}{y}dx = 0$?

Solve the D.E, $\frac{dy}{dx} = \frac{x\sqrt{1+y^2}}{2-3x^2}$?

2.2Homogeneous Equations

A first-order D.E in the form

$f(x,y)dx + g(x,y)dy = 0$ ………… (3)

Is a homogeneous type if both functions f(x,y) and g(x,y) are homogeneous of the same degree n. that is, multiplying each variable by a parameter λ, we find:

$f(\lambda x, \lambda y) = \lambda^n f(x,y)$ And $g(\lambda x, \lambda y) = \lambda^n g(x,y)$ … … … (4)

Thus,

$$\frac{f(\lambda x, \lambda y)}{g(\lambda x, \lambda y)} = \frac{f(x,y)}{g(x,y)} \text{ ……… } (5)$$

Equation (3) can be written in the form:

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = F(\frac{y}{x}) \ldots\ldots\ldots (6)$$

To solve eq. (6), put

$v = \frac{y}{x}$, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ $\ldots\ldots\ldots (7)$

Then, substitute eq. (7) in eq. (6) gives:

$$v + x\frac{dv}{dx} = F(v)$$

$$F(v) - v = x\frac{dv}{dx}$$

$\frac{dv}{F(v)-v} = \frac{dx}{x}$ … … … Separable D.E

By integrating both sides, we get the final solution

$$\int \frac{dv}{F(v) - v} = \ln(x) + c$$

Example 7: Find the general solution of the following D.E,

$$(x^3 + y^3)dx - 3xy^2dy = 0 \quad ?$$

Solution:

$$\frac{dy}{dx} = \frac{(x^3 + y^3)}{3xy^2}$$

Multiply by parameter λ

$$\frac{dy}{dx} = \frac{(\lambda^3x^3 + \lambda^3y^3)}{3\lambda x(\lambda y)^2} = \frac{\lambda^3(x^3 + y^3)}{\lambda^3 3xy^2} = \frac{(x^3 + y^3)}{3xy^2}$$

$\therefore \frac{f(\lambda x,\lambda y)}{g(\lambda x,\lambda y)} = \frac{f(x,y)}{g(x,y)}$ The D.E is homogeneous.

Dividing by x^3gives: (to get$\frac{dy}{dx} = F(\frac{y}{x})$)

$$\frac{dy}{dx} = \frac{1+(\frac{y}{x})^3}{3(\frac{y}{x})^2}$$

Assume $v = \frac{y}{x}$, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{1+(v)^3}{3(v)^2}$… … … Separable D.E

$$x\frac{dv}{dx} = \frac{1-2v^3}{3v^2}$$

$$\int \frac{3v^2}{1-2v^3} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2} ln(1-2v^3) = ln(x) + c$$

$\therefore ln\left(1-2\left(\frac{y}{x}\right)^3\right) + ln(x) = c_1$ Where $c_{1=-2C}$

Example 8: Solve the D.E,

$$(xe^{\frac{y}{x}} + y)dx - xdy = 0 \quad ?$$

Solution:

$$\frac{dy}{dx} = \frac{\left(xe^{\frac{y}{x}} + y\right)}{x}$$

Multiply by parameter λ

$$\frac{dy}{dx} = \frac{\left(\lambda\, xe^{\frac{\lambda\, y}{\lambda\, x}} + \lambda\, y\right)}{\lambda\, x} = \frac{\lambda\left(xe^{\frac{y}{x}} + y\right)}{\lambda x} = \frac{\left(xe^{\frac{y}{x}} + y\right)}{x}$$

$\therefore \dfrac{f(\lambda x, \lambda y)}{g(\lambda x, \lambda y)} = \dfrac{f(x,y)}{g(x,y)}$ The D.E is homogeneous.

Dividing by x gives:

$$\frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x}$$

Let $v = \frac{y}{x}$, $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = e^{v} + v \ldots \ldots \ldots$ Separable D.E

$$x\frac{dv}{dx} = e^{v} + v - v$$

$$\int \frac{dv}{e^{v}} = \int \frac{dx}{x}$$

$$-e^{-v} = \ln(x) + c$$

$$\therefore \ln(x) + e^{-(\frac{y}{x})} = c$$