TRIGONOMETRIC FUNCTIONS

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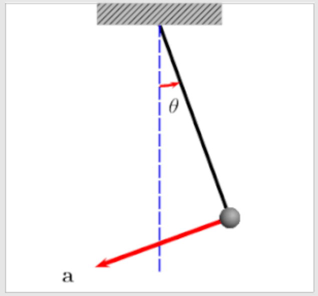
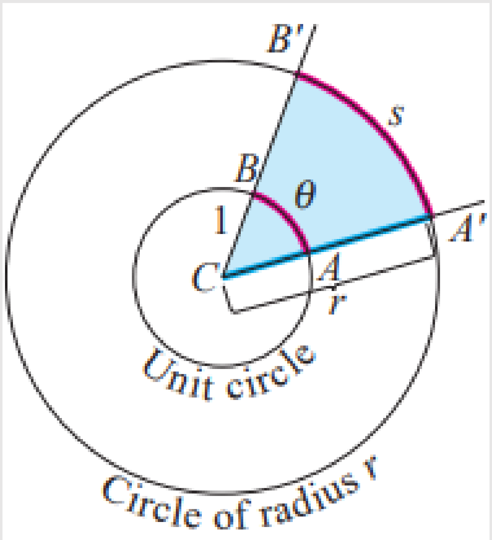
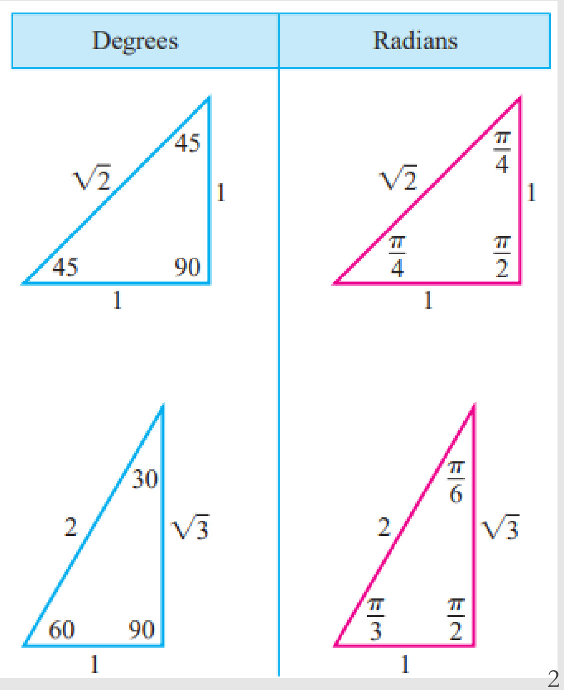
The trigonometric functions are important because they are periodic, or repeating, and therefore model many naturally occurring periodic processes.





Radian Measure

The radian measure of the angle ACB at the center of the unit circle (Figure) equals the length of the arc that ACB cuts from the unit circle. Figure shows that $s = r\theta$ is the length of arc cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians. Since the circumference of the circle is and one complete revolution of a circle is 360°, the relation between radians and degrees is given by



Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by

$$\pi \ radian = 180°$$

For example, 45° in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4}\ rad,$$

And $\frac{\pi}{6}$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°$$

Conversion Formulas

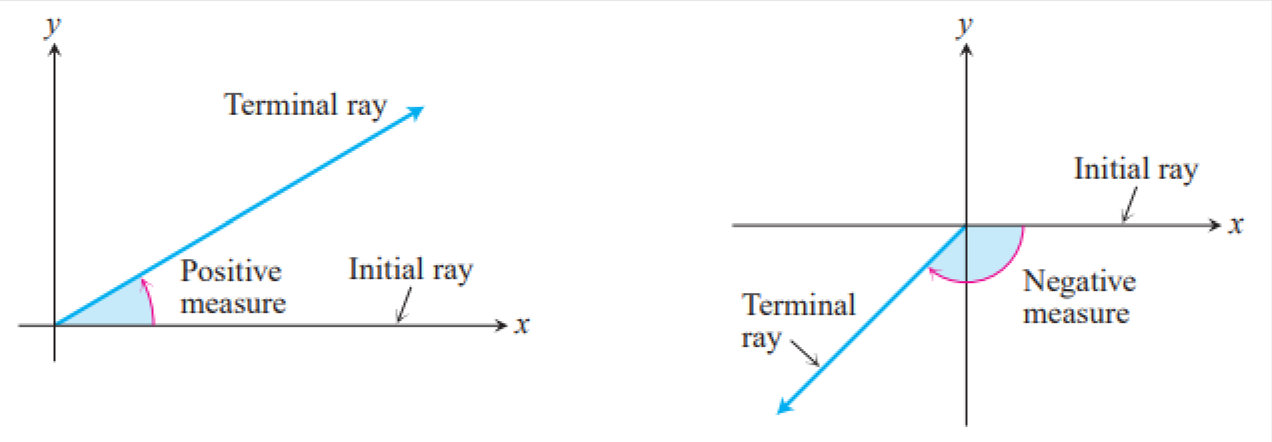
$$1 \text{ degree} = \frac{\pi}{180} (\approx 0.02) \text{ radians}$$

Degrees to radians: multiply by $\frac{\pi}{180}$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57) \text{ degrees}$$

Radians to degrees: multiply by $\frac{180}{\pi}$

An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Figure). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures



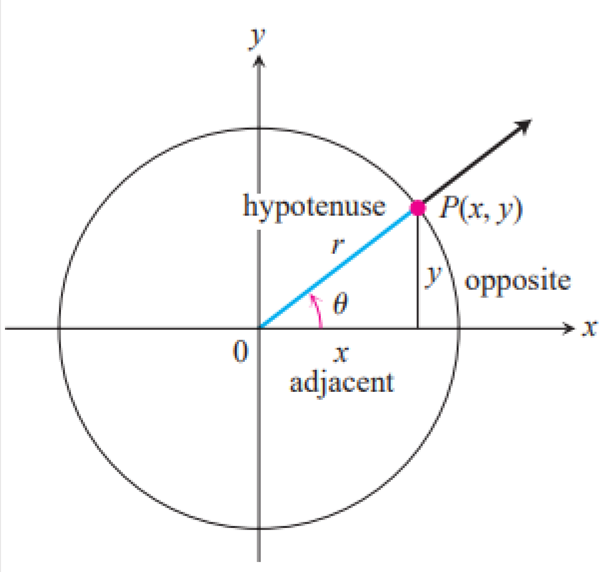
The Six Basic Trigonometric Functions

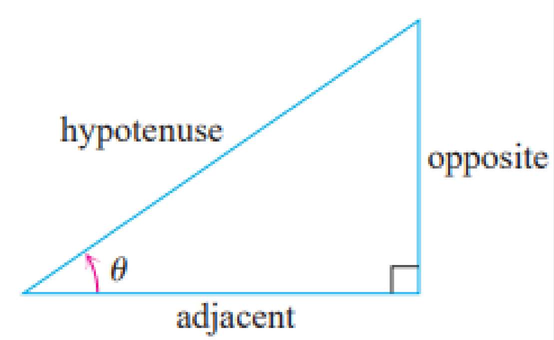
We then define the trigonometric functions in terms of the coordinates of the point P(x, y) where the angle's terminal ray intersects the circle.

sine: $\sin\theta = \dfrac{y}{r}$ **cosecant:** $\csc\theta = \dfrac{r}{y}$

cosine: $\cos\theta = \dfrac{x}{r}$ **secant:** $\sec\theta = \dfrac{r}{x}$

tangent: $\tan\theta = \dfrac{y}{x}$ **cotangent:** $\cot\theta = \dfrac{x}{y}$





$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\cot\theta = \dfrac{\text{adj}}{\text{opp}}$

The CAST rule (Figure) is useful for remembering when the basic trigonometric functions are positive or negative. For instance, from the triangle in Figure, we see that

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

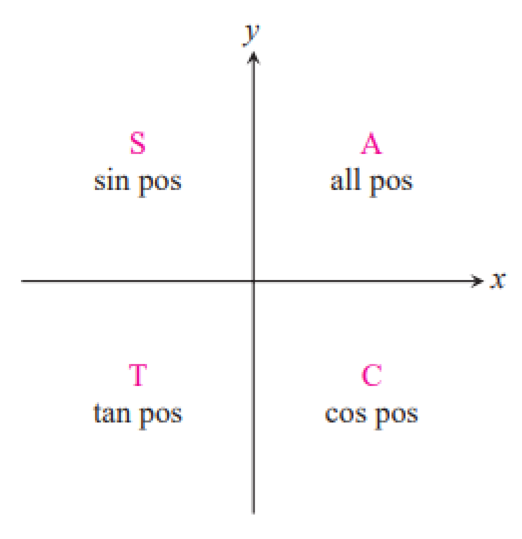
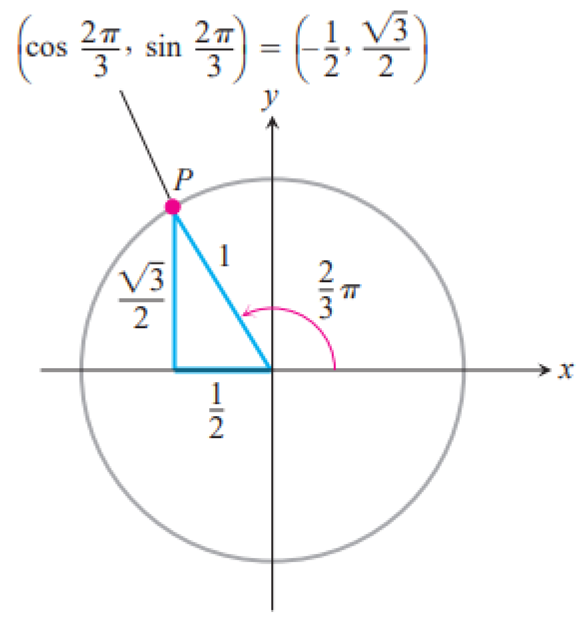


TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0



Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$\cos(\theta + 2\pi) = \cos\theta$	$\sin(\theta + 2\pi) = \sin\theta$	$\tan(\theta + 2\pi) = \tan\theta$
$\sec(\theta + 2\pi) = \sec\theta$	$\csc(\theta + 2\pi) = \csc\theta$	$\cot(\theta + 2\pi) = \cot\theta$

Similarly, $\cos(\theta - 2\pi) = \cos\theta, \sin(\theta - 2\pi) = \sin\theta$ and so on. The six basic trigonometric functions are periodic

DEFINITION Periodic Function

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

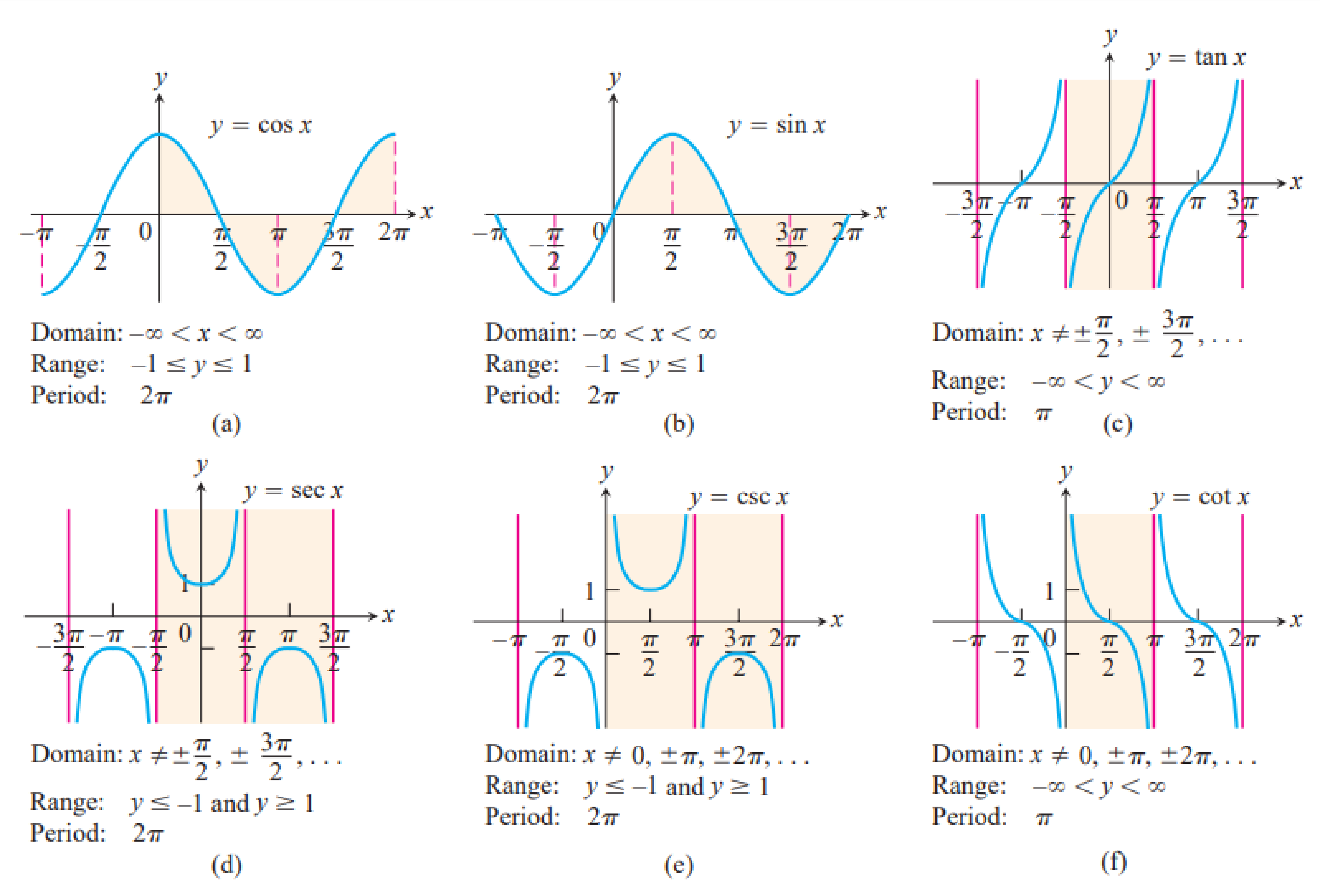
Periods of Trigonometric Functions

Period π:
$\tan(x + \pi) = \tan x$
$\cot(x + \pi) = \cot x$

Period 2π:
$\sin(x + 2\pi) = \sin x$
$\cos(x + 2\pi) = \cos x$
$\sec(x + 2\pi) = \sec x$
$\csc(x + 2\pi) = \csc x$

The symmetries in the graphs in Figure reveal that the cosine and secant functions are even and the other four functions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$


y = cos x
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(a)
y = sin x
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π
(b)
y = tan x
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π
(c)
y = sec x
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(d)
y = csc x
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π
(e)
y = cot x
Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π
(f)

Identities

The coordinates of any point P(x, y) in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Figure). Since $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$ we have

$$x = r\cos\theta, \qquad y = r\sin\theta$$

When r=1 we can apply the Pythagorean theorem to the reference right triangle in Figure and obtain the equation

$$\cos^2\theta + \sin^2\theta = 1. \qquad (1)$$

$$1 + \tan^2\theta = \sec^2\theta.$$

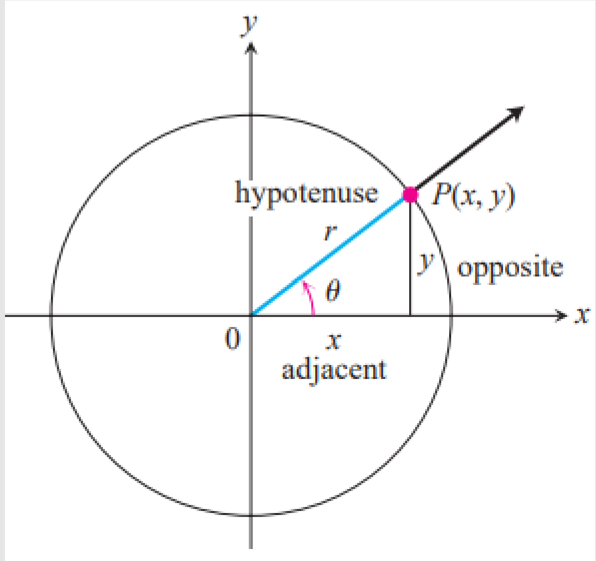
$$1 + \cot^2\theta = \csc^2\theta.$$

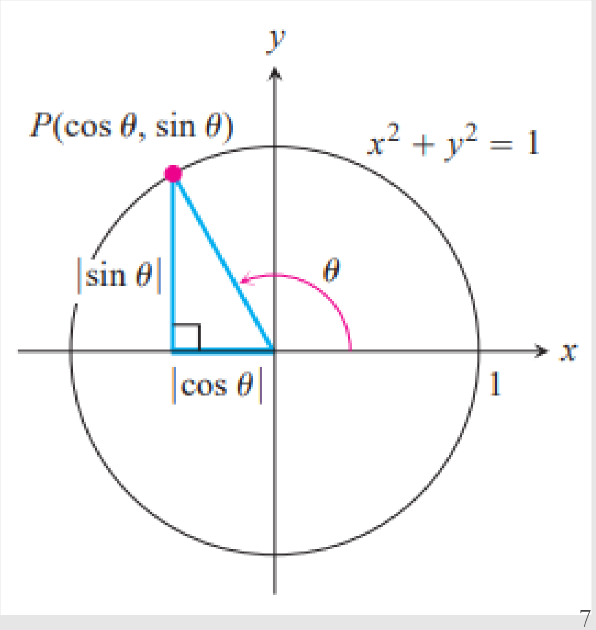
Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad (2)$$

There are similar formulas for and $\cos(A - B)$ and $\sin(A - B)$.





Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad (3)$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad (4)$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad (5)$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (6)$$

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.