Digital Signals Processing

معالجة الاشارات الرقمية

Lectures (Seven and Eight)

Subject: Digital Signals Processing

Lecture No.: Lectures Seven and Eight

Lecture Name: Analysis of Discrete Linear Time-Invariant Systems.

Pre test:

Circle the correct answer:

1. **Discrete time system is a system deals with:-**

   a- Continuous time signals.

   b- Discrete time signals.

   c- Digital signals.

   d- Speed.

2. **The output signals from discrete time systems may be defined as:**

a- Excitation. b- Noise.

c- Response. d- Any one of above.

3. **The input signals applied to discrete time systems may be defined as:**

a- Excitation. b- Noise.

c- Response. d- Any one of above.

4. **Analysis of discrete time systems needed for:-**

a- Rejected input signals.

b- Determine the response.

c- Determine the system's characteristics.

d- Any of above.

Difference Equations and Impulse Responses

Format of Difference Equation

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M), \quad (1)$$

where $a_1, \ldots, a_N$ and $b_0, b_1, \ldots,$ b are the coefficients of the difference equation. Equation (1) can further be written as

$$\begin{aligned} y(n) = & -a_1 y(n-1) - \ldots - a_N y(n-N) \\ & + b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \end{aligned} \quad (2)$$

Or

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j). \quad (3)$$

Notice that y(n) is the current output, which depends on the past output samples y(n _ 1), . . . , y(n _ N), the current input sample x(n), and the past input samples, x(n _ 1), . . . , x(n _ N).

We will examine the specific difference equations in the following examples.

Example

Given the following difference equation:

$$y(n) = 0.25y(n-1) + x(n),$$

Identify the nonzero system coefficients.

Solution:

Comparison with Equation (2) leads to

$$b_0 = 1$$
$$-a_1 = 0.25,$$

System Representation Using Its Impulse Response:

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input δ(n) with zero initial conditions, depicted in Figure (1).

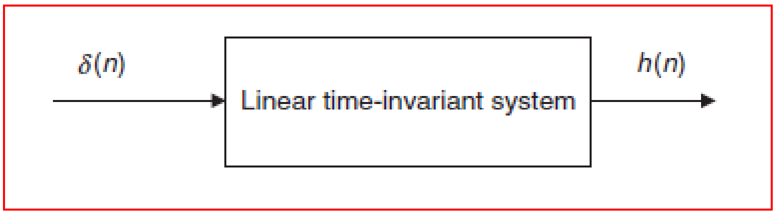


Figure (1): Unit-impulse response of the linear time-invariant system.

With the obtained unit-impulse response h(n), we can represent the linear time-invariant system in Figure (2).

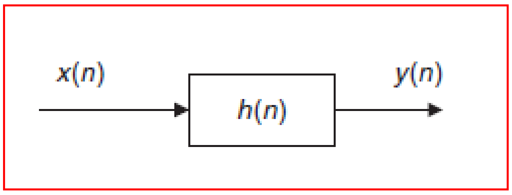


Figure (2): Representation of a linear time-invariant system using the impulse response.

Example:

Given the linear time-invariant system

$y(n) = 0.5x(n) + 0.25x(n - 1)$ with an initial condition $x(-1) = 0$,

a) Determine the unit-impulse response $h(n)$.
b) Draw the system block diagram.
c) Write the output using the obtained impulse response.

Solution:

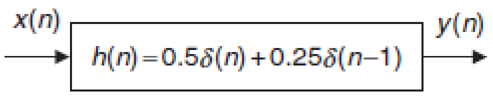
a) According to Figure 1, let $x(n) = \delta(n)$, then

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b) The block diagram of the linear time-invariant system is shown as



c) The system output can be rewritten as

$$y(n) = h(0)x(n) + h(1)x(n-1).$$

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as

$$y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots. \quad \textbf{(3)}$$

Equation (3) is called the digital convolution sum, which will be explored in a later section. We can verify Equation (3) by substituting the impulse sequence $x(n) = \delta(n)$ to get the impulse response

$$h(n) = \ldots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \ldots,$$

Example:

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

a) Determine the unit-impulse response *h(n).*
b) Draw the system block diagram.
c) Write the output using the obtained impulse response.
d) For a step input *x(n)* = *u(n),* verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3).

Solution:

a) Let *x(n)* = *δ(n),* then

$$h(n) = 0.25h(n-1) + \delta(n).$$

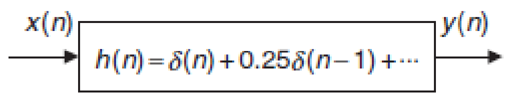
To solve for *h(n),* we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$
$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.25 + 0 = 0.0625$$

With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots.$$

b) The system block diagram is given in Figure below.



c) The output sequence is a sum of infinite terms expressed as

$$\begin{aligned} y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \\ &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \end{aligned}$$

d) From the difference equation and using the zero-initial condition, we have

$$\begin{aligned} &y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0 \\ &n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1 \\ &n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25 \\ &n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125 \\ &\ldots. \end{aligned}$$

Applying the convolution sum in Equation (3) yields

$$y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots$$

$$\begin{aligned} n = 0,\ y(0) &= x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots \\ &= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1 \end{aligned}$$

$$\begin{aligned} n = 1,\ y(1) &= x(1) + 0.25x(0) + 0.0625x(-1) + \ldots \\ &= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25 \end{aligned}$$

$$\begin{aligned} n = 2,\ y(2) &= x(2) + 0.25x(1) + 0.0625x(0) + \ldots \\ &= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125 \end{aligned}$$

Digital Convolution

Given a linear time-invariant system, we can determine its unit-impulse response *h(n),* which relates the system input and output. To find the output sequence *y(n)* for any input sequence *x(n)*, we write the digital convolution as shown in Equation (3) as:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \end{aligned}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots \quad \textbf{(4)}$$

Using a conventional notation, we express the digital convolution as

$$y(n) = h(n) * x(n). \quad \textbf{(5)}$$

Note that for a causal system, which implies its impulse response

$$h(n) = 0 \text{ for } n < 0,$$

The lower limit of the convolution sum begins at 0 instead of 1, that is

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k).$$

We will focus on evaluating the convolution sum based on Equation (4). Let us examine first a few outputs from Equation (4):

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = \ldots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + x(2)h(-2) + \ldots$$
$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \ldots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1) + \ldots$$
$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \ldots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) + \ldots$$
$$\ldots$$

Digital convolution using the graphical method.
Step 1. Obtain the reversed sequence $h(-k)$.
Step 2. Shift $h(-k)$ by $\lvert n \rvert$ samples to get $h(n-k)$. If $n \geq 0$, $h(-k)$ will be shifted to the right by n samples; but if $n < 0$, $h(-k)$ will be shifted to the left by $\lvert n \rvert$ samples.
Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and $h(n-k)$ to get $y(n)$.
Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Example

Given a sequence,

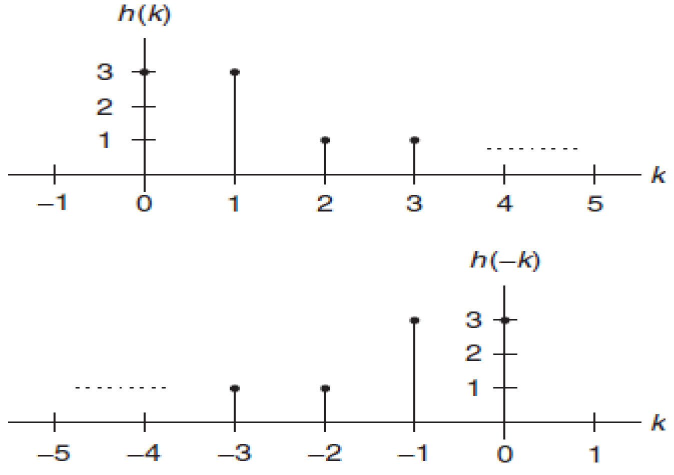
$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

where k is the time index or sample number,

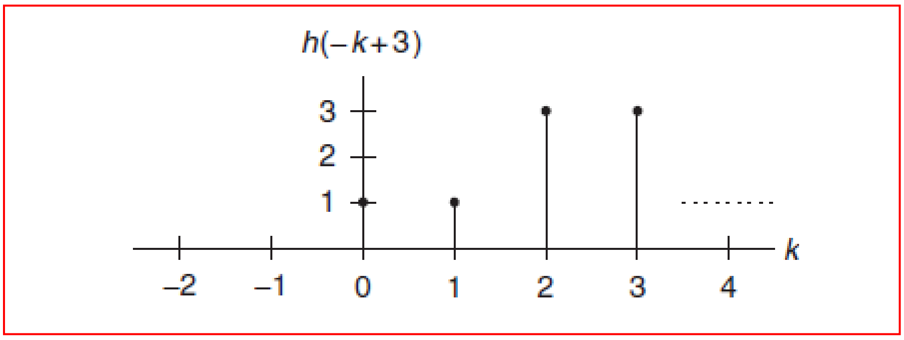
a) Sketch the sequence *h(k)* and reversed sequence h(- k).
b) Sketch the shifted sequences *h(-k +3)* and *h(- k - 2)*.

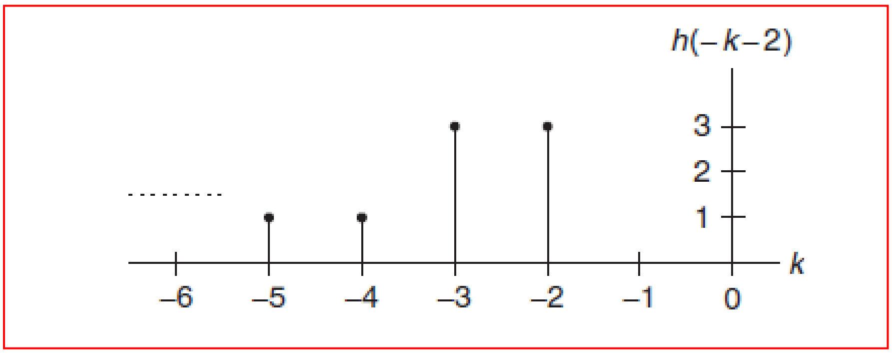
Solution:

a)



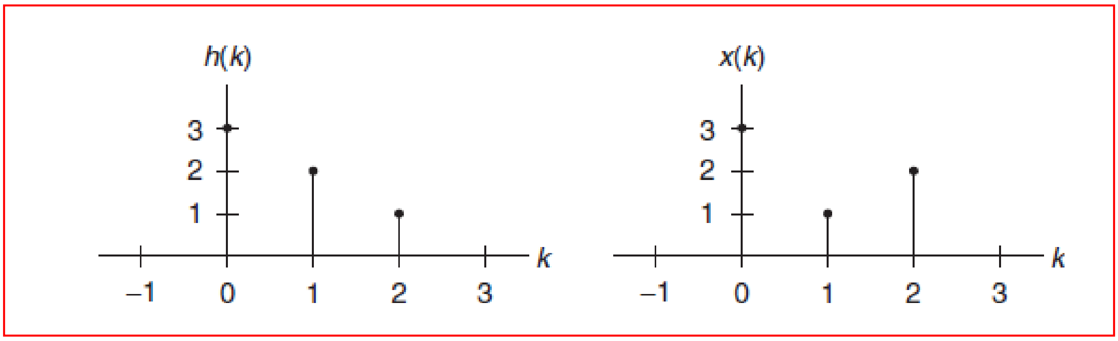
b)





Example

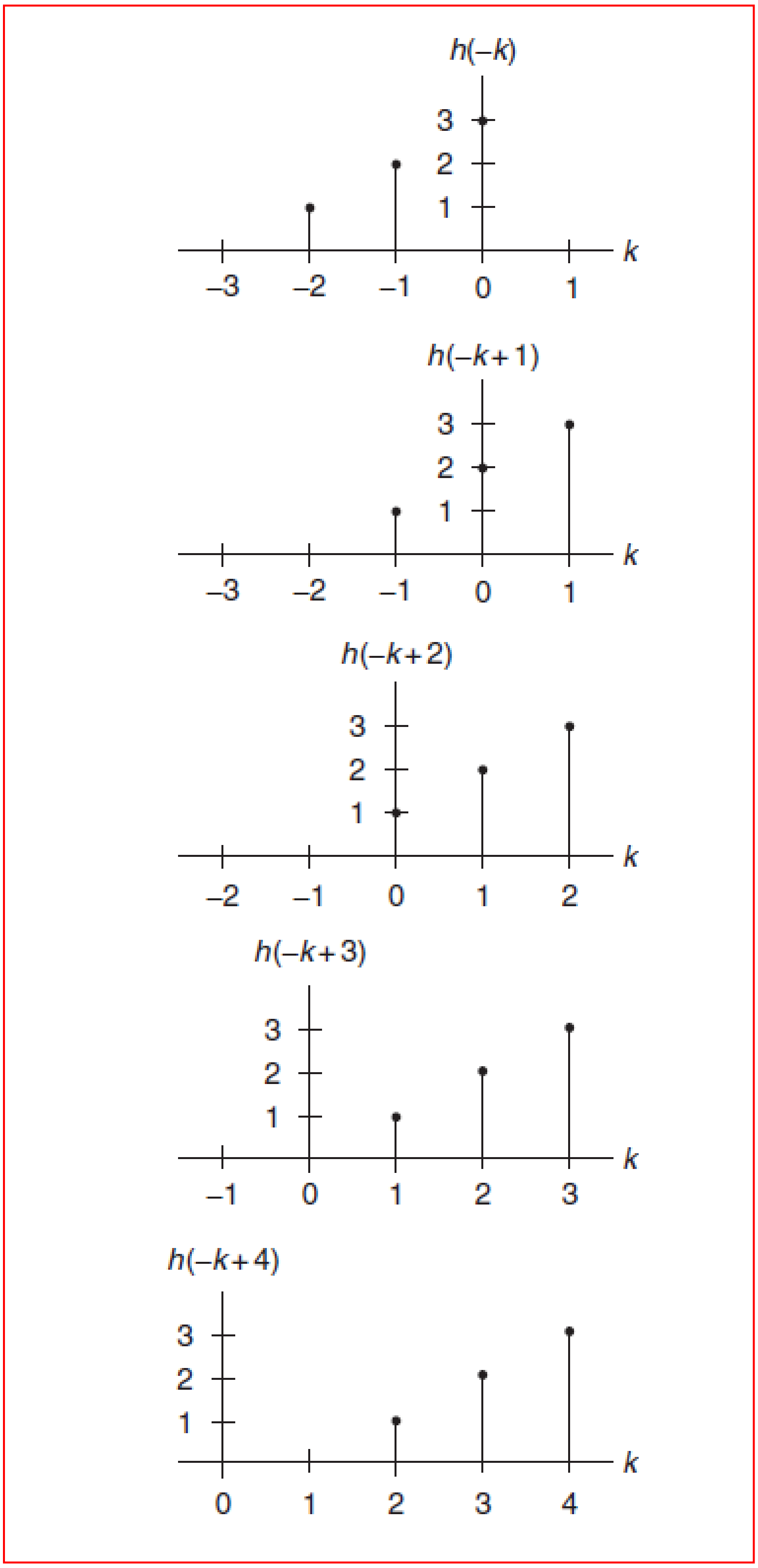
Using the following sequences defined in the Figure below, evaluate the digital convolution



a) By the graphical method.
b) By applying the formula directly.

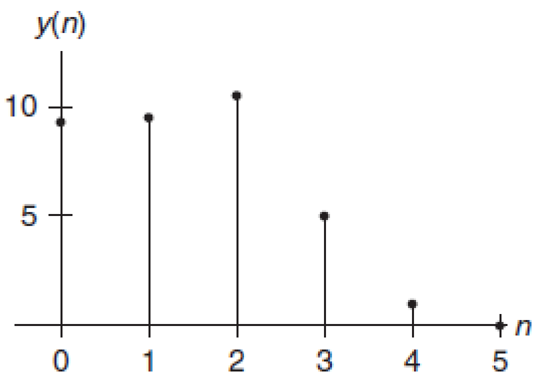
Solution:

a)


h(−k)
3
2
1
k
−3
−2
−1
0
1
h(−k+1)
3
2
1
k
−3
−2
−1
0
1
h(−k+2)
3
2
1
k
−2
−1
0
1
2
h(−k+3)
3
2
1
k
−1
0
1
2
3
h(−k+4)
3
2
1
k
0
1
2
3
4

We can compute the convolution sum as:

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.



b) Applying Equation (4) with zero initial conditions leads to

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

Example:

Given the following two rectangular sequences,

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & otherwise \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

Convolve them using the table method.

Solution:

Digital convolution steps via the table.

Step 1. List the index k covering a sufficient range.

Step 2. List the input $x(k)$.

Step 3. Obtain the reversed sequence $h(-k)$, and align the rightmost element of $h(n-k)$ to the leftmost element of $x(k)$.

Step 4. Cross-multiply and sum the nonzero overlap terms to produce $y(n)$.

Step 5. Slide $h(n-k)$ to the right by one position.

Step 6. Repeat step 4; stop if all the output values are zero or if required.

Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Post test:

1. A causal, linear, discrete time-invariant system can be described by:
   a) Digital numbers.
   b) Algebraic equation.
   c) Differential equations.
   d) Difference equations.
2. Impulse response of discrete time systems defined as:
   a) System response to unit step input.
   b) System response to unit impulse input.
   c) System response to exponential input.
   d) System response to arbitrary input.
3. Digital convolution determines:
   a) System response to any input by using impulse response.
   b) System response to any input by using step response.
   c) System response to any input by using exponential response.
   d) Non of the above.
4. Digital convolution can be determined using:
   a) Graphical methods.
   b) Mathematical methods.
   c) Tables.
   d) Any of the above.

Key Answers

Pre test:

1.b **2.c** **3.a** **4.c**

Post test:

1.d **2.b** **3.a** **4.d**