Reducible to Homogenous

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0$$

$a_1, a_2, b_1, b_2, c_1, c_2$ are constant

ex/ solve the D.E $(x+y+1)\,dy=(x+y-5)\,dx$

sol/ $\frac{dy}{dx}=\frac{x+y-5}{x+y+1}$ let $z=x+y$

$$\frac{dz}{dx}=1+\frac{dy}{dx} \longrightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$$

$$\frac{dz}{dx}-1=\frac{z-5}{z+1} \longrightarrow \frac{dz}{dx}=\frac{z-5}{z+1}+1$$

$$\frac{dz}{dx}=\frac{2z-4}{z+1}$$ by sparable

$$\int\frac{dz}{\frac{2z-4}{z+1}}=\int\frac{dx}{1} \longrightarrow \int\frac{z+1}{2z-4}\,dz=\int dx$$

$$\int\frac{1}{2}\,dz+^{\frac{2}{2}*}\int\frac{3}{2z-4}\,dz=\int dx$$

$$\frac{1}{2}z+\frac{3}{2}\ln(2z-4)=x+c$$

Reducible to exact

when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ use thies Method

$$f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \underline{\underline{or}} \quad f(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

ex/ Solve the D.E $(x^2+y^2+x)dx + xy\,dy = 0$

sol/ $M = x^2+y^2+x \rightarrow \frac{\partial M}{\partial y} = 2y$

$N = xy \longrightarrow \frac{\partial N}{\partial x} = y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

$$f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{\cancel{y}}{x\cancel{y}} \rightarrow \frac{1}{x}$$

$$IF = e^{\int P(x)\,dx} \rightarrow e^{\int \frac{1}{x} dx} \rightarrow \cancel{e}^{\cancel{\ln} x} = \boxed{x}$$

$\therefore (x^2+y^2+x)dx + xy\,dy = 0 \,\} * x$

$(x^3+xy^2+x^2)dx + x^2y\,dy = 0$

$M = x^3+xy^2+x^2 \longrightarrow \frac{\partial M}{\partial y} = 2xy$ $\therefore$ exact

$N = x^2y \longrightarrow \frac{\partial N}{\partial x} = 2xy$

$$M = \frac{\partial F}{\partial x} \longrightarrow \partial F = M\,\partial x \longrightarrow \int \partial F = \int M\,\partial x$$

$$F = \int x^3 + xy^2 + x^2 \quad dx \longrightarrow F = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + C_y$$

$$\because \frac{\partial f}{\partial y} = N \Longrightarrow \cancel{x^2 y} = \cancel{x^2 y} + \frac{\partial c_y}{\partial y}$$

$$\therefore \int \partial c_y = \int \partial y \longrightarrow C_y = C$$

---

ex/solu $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{not exact}$$

$$g(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

$$= \frac{(8xy^3e^y + 2xy^4e^y + 6xy^2 + 1) - (2xy^4e^y - 2xy^2 - 3)}{(2xy^4e^y + 2xy^3 + y)}$$

$$= \frac{8xy^3e^y + 8xy^2 + 4}{2xy^4e^y + 2xy^3 + y} \longrightarrow \frac{4(\cancel{2xy^3e^y + 2xy^2 + 1})}{y(\cancel{2xy^3e^y + 2xy^2 + 1})} = \frac{4}{y}$$

$$IF = e^{-\int \frac{4}{y} dy} \rightarrow e^{-4\ln y} \rightarrow \cancel{e}^{\ln y^{-4}} \rightarrow \frac{1}{y^4}$$

$$\left(2xy^4 e^y * \frac{1}{y^4} + 2x\frac{y^3}{y^4} + \frac{y}{y^4}\right)dx + \left(\frac{x^2y^4e^y}{y^4} - \frac{x^2y^2}{y^4} - 3\frac{x}{y^4}\right)dy$$

$$\left(2xe^y + 2x\frac{1}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\frac{\partial N}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

exact

$$\because M = \frac{\partial F}{\partial x} \rightarrow \partial F = M\partial x \rightarrow \int \partial F = \int M \partial x$$

$$F = \int 2xe^y + 2x\frac{1}{y} + \frac{1}{y^3} \partial x$$

$$F = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} + C_y \quad —①$$

$$\because \frac{\partial F}{\partial y} = N \rightarrow$$

$$\cancel{x^2 e^y} - \cancel{\frac{x^2}{y^2}} - 3\cancel{\frac{x}{y^4}} + \frac{\partial C_y}{\partial y} = \cancel{x^2e^y} - \cancel{\frac{x^2}{y^2}} - \cancel{\frac{3x}{y^4}}$$

$$\partial C_y = \partial y \rightarrow \int \partial C_y = \int \partial y \rightarrow \therefore C_y = C \quad —②$$

اعوض في ①

$$\therefore F = x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} + C$$

④

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second order D.E

1. Homogeneous 2^{nd}-order D.E

$$a y'' + b y' + c y = 0$$

Sol/ $y = e^{mx} \longrightarrow y' = m e^{mx} \longrightarrow y'' = m^2 e^{mx}$

$$a m^2 e^{mx} + b m e^{mx} + c e^{mx} = 0$$

$$(a m^2 + b m + c) e^{mx} = 0$$

$$\therefore \ a m^2 + b m + c = 0$$

$$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

① If $m_1 \neq m_2 \longrightarrow y_h = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

② if $m_1 = m_2 = m \longrightarrow y_h = C_1 e^{mx} + C_2 e^{mx}$

③ if $\left.\begin{matrix} m_1 = \alpha + \beta i \\ m_2 = \alpha - \beta i \end{matrix}\right\} \longrightarrow y_h = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x)$

جذر خيالي $i = \sqrt{-1}$

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ex/ $y'' + 4y' + 3y = 0$

$y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$\therefore m^2 e^{mx} + 4me^{mx} + 3e^{mx} = 0$

$(m^2 + 4m + 3)e^{mx} = 0$

$(m + 3)(m + 1) = 0$

$\therefore m_1 = -3 \quad m_2 = -1$

$y_h = c_1 e^{-3x} + c_2 e^{-x}$

ex/ $y'' - 6y' + 9y = 0$

$y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} - 6me^{mx} + 9e^{mx} = 0$

$(m^2 - 6m + 9)e^{mx} = 0$

$(m - 3)(m - 3) = 0$

$\therefore m_1 = m_2 = 3$

$y_h = c_1 e^{3x} + c_2 e^{3x}$

ex/ $y'' + 3y' - 10y = 0$

$y = e^{mx} \rightarrow y' = me^{mx} \rightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} + 3me^{mx} - 10e^{mx} = 0$

$(m+5)(m-2) = 0$

$\therefore \quad m_1 = -5 \quad , m_2 = 2$

$y_h = c_1 e^{-5x} + c_2 e^{2x}$

ex/ $y'' + 2y' + 5y = 0$

$y = e^{mx} \longrightarrow y' = me^{mx} \longrightarrow y'' = m^2 e^{mx}$

$m^2 e^{mx} + 2me^{mx} + 5e^{mx} = 0$

$m^2 + 2m + 5 = 0$

$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \mp \sqrt{4-20}}{2}$

$= -1 \mp \frac{\sqrt{-16}}{2} \Longrightarrow \frac{-1 \mp \sqrt{4-20}}{2}$

$= -1 \mp \frac{\sqrt{-16}}{2} \longrightarrow -1 \mp \frac{4}{2}\sqrt{-1}$

$= -1 \mp 2i$

Root are $\quad m_1 = -1 + 2i \quad\quad m_2 = -1 - 2i \quad\quad \alpha = -1 \quad \beta = 2$

$y_h = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$

second order D.E

2. Non-Homogeneous 2^{nd}-order D.E

$$ay'' + by' + cy = f(x)$$

$$y_g = y_h + y_p$$

y_g : general Solution

y_h : homogeneous sol. $(ay'' + by' + cy = 0)$

y_p : Partical sol.

y_p

- undetermined cofficients method
- variation of Parameters

$f(x)$	Note	y_p
constant		a_0
Ae^{ax}	if (a) is not root of eq.	$a_0 e^{ax}$
	" one root "	$a_0 x e^{ax}$
	" repeated root "	$a_0 x^2 e^{ax}$
Polynomial ax^3+bx^2+cx+d	if is not root of eq.	$a_0+a_1x+a_2x^2+a_3x^3$
	" one root of eq.	$x(a_0+a_1x+a_2x^2+a_3x^3)$
	" repeated root "	$x^2(a_0+a_1x+a_2x^2+a_3x^3)$
$A\cos\beta x$ $A\sin\beta x$	if (βi) is not root of eq	$a_0\sin\beta x+a_1\cos\beta x$
	" one root of eq.	$x(a_0\sin\beta x+a_1\cos\beta x)$

* يجب ايجاد الثوابت a_0, a_1, a_2 وذلك عن طريق اشتقاق y_p $\frac{dy_p}{dx}$, $\frac{d^2y_p}{dx^2}$ وتعويضهم في المعادلة التفاضلية

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ex/ solve $y'' + 2y' - 3y = 6$

$y_g = y_h + y_p$

① $y_h \Rightarrow$ ايجاد

$y'' + 2y' - 3y = 0$

$y = e^{mx}$, $y' = me^{x}$, $y'' = m^2 e^{mx}$

$m^2 e^{mx} + 2m e^{mx} - 3e^{mx} = 0$

$(m^2 + 2m - 3)e^{mx} = 0$

The chara. eq. is $m^2 + 2m - 3 = 0$

The roots are $(m+3)(m-1) = 0$

$\therefore m_1 = -3 \qquad m_2 = 1$

$y_h = c_1 e^{-3x} + c_2 e^{3x}$

② $y_p \Rightarrow$ ايجاد

$f(x) = 6$

$y_p = a_0 \longrightarrow \frac{dy_p}{dx} = 0 \longrightarrow \frac{d^2 y_p}{dx^2} = 0$

$0 + 2(0) - 3a_0 = 6$

$\therefore a_0 = -2 \longrightarrow \therefore y_p = -2$

The general solution $y = C_1 e^{-3x} + C_2 e^{3x} - 2$

ex / solve $y'' - 6y' + 9y = 2e^{3x}$

$y_g = y_h + y_p$

① $y_h \rightarrow$ ايجاد

$y'' - 6y' + 9y = 0 \rightarrow \therefore y = e^{mx}, \; y' = me^{mx}, \; y'' = m^2 e^{mx}$

$m^2 e^{mx} - 6me^{mx} + 9e^{mx} = 0 \longrightarrow (m^2 - 6m + 9)e^{mx} = 0$

The chara eq. is $m^2 - 6m + 9 = 0 \longrightarrow (m-3)(m-3) = 0$

$\therefore$ The root $m_1 = m_2 = 3$

$y_h = (c_1 + x c_2) e^{3x}$

② $y_p \rightarrow$ ايجاد $\quad f(x) = 2e^{3x}$

$y_p = a_0 x^2 e^{3x} \quad , \; y' = a_0 x^2 * 3e^{3x} + e^{3x} * 2x a_0$

$y'' = 6a_0 x * e^{3x} + 9e^{3x} * a_0 x^2 + 2a_0 e^{3x} + 6e^{3x} * a_0 x$

* اعوض y, y', y'' في المعادلة الرئيسية

$(6a_0 x e^{3x} + 9e^{3x} a_0 x^2 + 2a_0 e^{3x} + 6e^{3x} a_0 x) -$

$6(a_0 x^2 \cdot 3e^{3x} + 2x a_0 e^{3x}) + 9a_0 x^2 e^{3x} = 2e^{3x}$

$9a_0 x^2 e^{3x} + 12a_0 x e^{3x} + 2a_0 e^{3x} - 18a_0 x^2 e^{3x} - 12a_0 x e^{3x} + 9a_0 x^2 e^{3x}$

$2a_0 e^{3x} = 2e^{3x} \longrightarrow 2a_0 = 2 \longrightarrow a_0 = 1$

$\therefore y_p = x^2 e^{3x}$

$\therefore y_g = (c_1 + x c_2) e^{3x} + x^2 e^{3x}$

ex/solve $y'' - 4y' + 4y = 4x + 8x^3$

$y_g = y_h + y_p$

① $y_h \longrightarrow$ ايجاد $\because y'' - 4y' + 4y = 0$

$y = e^{mx} \; , \; y' = me^{mx} \; , \; y'' = m^2 e^{mx}$

$m^2 e^{mx} - 4me^{mx} + 4e^{mx} = 0 \longrightarrow (m^2 - 4m + 4)e^{mx} = 0$

$(m-2)(m-2) = 0 \longrightarrow \therefore m_1 = m_2 = 2$

$\therefore y_h = c_1 e^{2x} + c_2 x e^{2x}$

② $y_p \longrightarrow$ ايجاد $\because f(x) = 4x + 8x^3$ Polynomail

$\therefore y_p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$y'_p = a_1 + 2a_2 x + 3a_3 x^2$

$y''_p = 2a_2 + 6a_3 x$

y
y'
y''
اعوض في المعادلة الرئيسية

$(2a_2 + 6a_3 x) - 4(a_1 + 2a_2 x + 3a_3 x^2) + 4(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = 4x + 8x^3$

$\therefore 2a_2 + 6a_3 x - 4a_1 - 8a_2 x - 12a_3 x^2 + 4a_0 + 4a_1 x + 4a_2 x^2 + 4a_3 x^3 = 4x + 8x^3$

$4a_3 x^3 + (4a_2 - 12a_3)x^2 + (6a_3 - 8a_2 + 4a_1)x + (2a_2 - 4a_1 + 4a_0) = 4x + 8x^3$

$4a_3x^3 = 8x^3 \longrightarrow \therefore a_3 = 2$

$4a_2 - 12a_3 = 0 \longrightarrow 4a_2 - 12(2) = 0 \longrightarrow \therefore a_2 = 6$

$6a_3 - 8a_2 + 4a_1 = 4 \longrightarrow 6(2) - 8(6) + 4a_1 = 4 \longrightarrow \therefore a_1 = 10$

$2a_2 - 4a_1 + 4a_0 = 0 \longrightarrow 2(6) - 4(10) + 4a_0 = 0 \longrightarrow \therefore a_0 = 7$

$\therefore y_p = 7 + 10x + 6x^2 + 2x^3$

$\therefore y_g = c_1 e^{2x} + c_2 x e^{2x} + 7 + 10x + 6x^2 + 2x^3$

Solve $y'' - 4y' - 5y = 2\sin 2x$

① $y_h \longrightarrow$ ايجاد $y'' - 4y' - 5y = 0$

$y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$(m^2 - 4m - 5)e^{mx} = 0 \longrightarrow (m-5)(m+1) = 0$

$\therefore m_1 = 5 \quad m_2 = -1$

$\therefore y_h = C_1 e^{5x} + C_2 e^{-1x}$

② $y_p \longrightarrow$ ايجاد $\because f(x) = 2\sin 2x$

$y_p = a_0 \sin 2x + a_1 \cos 2x$

$y'_p = 2a_0 \cos 2x - 2a_1 \sin 2x$

$y''_p = -4a_0 \sin 2x - 4a_1 \cos 2x$

$(-4a_0 \sin 2x - 4a_1 \cos 2x) - 4(2a_0 \cos 2x - 2a_1 \sin 2x) -$
$5(a_0 \sin 2x + a_1 \cos 2x) = 2\sin 2x$

$-4a_0 \sin 2x - 4a_1 \cos 2x - 8a_0 \cos 2x + 8a_1 \sin 2x - 5a_0 \sin 2x$
$- 5a_1 \cos 2x = 2\sin 2x$

$(-4a_0 + 8a_1 - 5a_0)\sin 2x + (-4a_1 - 8a_0 - 5a_1)\cos 2x$

$(8a_1 - 9a_0)\sin 2x + (-9a_1 - 8a_0)\cos 2x = 2\sin 2x$

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$\therefore \; 8a_1 - 9a_0 = 2 \quad —①$

$-9a_1 - 8a_0 = 0$

لحل هذه المعادلتين لايجاد قيم a_0 , a_1 بطريقة تعويض

$a_1 = \dfrac{-8}{9} a_0$ —— اعوضها في ①

$\dfrac{-64}{9} a_0 - \dfrac{9a_0}{1} = 2 \longrightarrow \dfrac{-64a_0 - 81a_0}{9} = 2$

$\therefore \; -145 a_0 = 18 \longrightarrow a_0 = \dfrac{-18}{145} \quad \therefore a_1 = \dfrac{16}{145}$

$\therefore \; y_p = \dfrac{-18}{145} \sin 2x + \dfrac{16}{145} \cos 2x$

$\therefore \; y_g = c_1 e^{5x} + c_2 e^{-x} - \dfrac{18}{145} \sin 2x + \dfrac{16}{145} \cos 2x$

2. Infinite Series

A series is the sum of a sequence of numbers.

$\left[a_n\right]_{n=1}^{\infty} = a_1, a_2, a_3, a_4 \ldots\ldots, a_n, \ldots \rightarrow$ Sequence

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots + a_n + \ldots \rightarrow$ Series

where, a_n is the nth term of the series.

The sequence $\{S_n\}$ defined by:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots\ldots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of the series, the number S_n being the nth partial sum.

Note

(1) If the sequence of partial sums converges to a limit L, where L is a real number, $\lim_{n \to \infty} S_n = L$

L total sum

S_n partial sum

then, the series $\sum a_n$ converges, and that its sum is L,

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

$\sum a_n$ → partial sum

L → total sum

(2) If the sequence of partial sum of the series does not converge, we say that the series diverges.

(3) The sigma notation (summation) is used to write the series as:

$$\sum_{n=1}^{\infty} a_n \quad , \quad \sum_{k=1}^{\infty} a_k \quad \text{or} \quad \sum a_n$$

Example ① Find The n-Th term or partial sum for

To find meaning to an expression like

$1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$

we add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

partial Sum	Value	Suggestive expression
①: $S_1 = 1$	1	$2-1$
②: $S_2 = 1 + 1/2$	$3/2$	$2 - 1/2$
③: $S_3 = 1 + 1/2 + 1/4$	$7/4$	$2 - 1/4$
⋮	⋮	⋮
nth: $S_n = 1 + 1/2 + 1/4 + \cdots + 1/2^{n-1}$	$\frac{2^n - 1}{2^{n-1}}$	$2 - \frac{1}{2^{n-1}}$

the partial sums for a sequence whose nth term is :

$$\boxed{S_n = 2 - \frac{1}{2^{n-1}}}$$

This sequence of partial sums converges to $\underline{\underline{2}}$ because

$$\lim_{n\to\infty} \left(\frac{1}{2^{n-1}}\right) = 0$$

∴ The sum of the infinite series

$$1 + 1/2 + 1/4 + \cdots\cdots + \frac{1}{2^{n-1}}$$

$$\sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} 2 - \frac{1}{2^{n-1}} = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right)$$

$$= \lim_{n\to\infty} 2 - \cancel{\lim_{n\to\infty} \frac{1}{2^{n-1}}}^{\,0}$$

$$= \underline{\underline{2}} \quad \text{(total sum)}$$

Example ②

Find this series converges or not ?

$$1/2 + 1/6 + 1/12 + \frac{1}{20} + \cdots + \frac{1}{n(n+1)} + \cdots$$

Solution

$$a_1 + a_2 + a_3 + \cdots\cdots + a_n + \cdots$$

partial Sum	Value	Suggestive expression
$S_1 = \frac{1}{1\times2}$ $\frac{1}{2}$	$\boxed{\frac{1}{2}}$	$1 - \frac{1}{2} = \frac{1}{2}$
$S_2 = \frac{1}{1\times2} + \frac{1}{2\times3}$	$\frac{2}{3}$	$1 - \frac{1}{3} = \frac{2}{3}$
$S_3 = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4}$	$\frac{3}{4}$	$1 - \frac{1}{4} = \frac{3}{4}$
⋮	⋮	⋮
$S_n = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{n(n+1)}$	$\frac{n}{(n+1)}$	$1 - \frac{1}{(n+1)} = \frac{n}{(n+1)}$

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∴ the nth partial sum $S_n = 1 - \frac{1}{n+1}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - \frac{1}{\infty} = 1 \quad \text{(Converges)}$$

Geometric Series

Geometric series are series of the form:-

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots$$

in which a and r are fixed real numbers and $a \neq 0$. (المقام ثابتة والأس متغير)

a --- is the first term.

$$r = \text{ratio} = \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \cdots$$

This series can also be written as

$$\sum_{n=0}^{\infty} ar^n .$$

The ratio r can be positive, as in:-

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots \qquad \left(a=1,\ r=\frac{1}{2}\right)$$

or negative, as in:-

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots \qquad \left(a=1,\ r=-\frac{1}{3}\right)$$

Notes

① If $r=1$, the nth partial sum of the geometric series is:

$$S_n = a + a(1) + a(1)^2 + a(1)^3 + a(1)^4 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} S_n = \mp\infty$

② If $|r|<1$, then the geometric series converges to $\frac{a}{(1-r)}$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$ partial sum

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}} \qquad |r|<1$$

③ If $|r|>1$, the series diverges.

Example ①

The geometric series with $\boxed{a=1/9}$ and $\boxed{r=1/3}$ is

$$1/9 + 1/27 + 1/81 + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \boxed{\frac{1}{6}}$$

Example ②

The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

is a geometric series with $a=5$ and $r=-\frac{1}{4}$.

It converges to:

$$\frac{a}{1-r} = \frac{5}{1-\left(\frac{-1}{4}\right)} = \boxed{4}$$

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Example ③

Test and find the summation if the series are converged?

① $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

$= \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots\cdots$

$\therefore a = \frac{3}{2}$

$r = \frac{9/4}{3/2} = \frac{3}{2}$

$\therefore |r| > 1$ (diverges) No summation.

② $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots$

$\therefore a = \frac{1}{2}$

$r = \frac{(1/2)^2}{(1/2)} = \frac{1}{2}$

$\therefore |r| < 1$ (Converges)

$\text{Sum} = \frac{a}{1-r} = \frac{(1/2)}{1-(1/2)} = \boxed{1}$

③ $\sum_{n=1}^{\infty} \frac{1}{3^n}$

$= \left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

$$\boxed{\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}}$$

$a = \frac{1}{3}$

$r = \frac{(1/9)}{(1/3)} = \frac{1}{3} \Longrightarrow |r| < 1$ (Converges)

$\therefore \text{Sum} = \frac{a}{1-r} = \frac{(1/3)}{1-(1/3)} = \frac{1}{2}$

④ $\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$

$= \left(\frac{\pi}{e}\right) + \left(\frac{\pi}{e}\right)^2 + \left(\frac{\pi}{e}\right)^3 + \cdots$

$\therefore\ a = \frac{\pi}{e}$

$r = \frac{(\pi/e)^2}{(\pi/e)} = \frac{\pi}{e} = \frac{3.14}{2.718}$

$\therefore\ |r| > 1$ (diverges) No summation

⑤ $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^{n-1}$

$= 1 + \left(\frac{e}{\pi}\right) + \left(\frac{e}{\pi}\right)^2 + \left(\frac{e}{\pi}\right)^3 + \cdots$

$\therefore\ a = 1$

$r = \frac{(e/\pi)}{1} = \frac{e}{\pi} = \frac{2.718}{3.14} = 0.865$

$\therefore\ |r| < 1$ (Converges)

$\therefore$ Sum $= \frac{a}{1-r} = \frac{1}{1-0.865} = 7.407$

⑥ $\sum_{n=1}^{\infty} \frac{(3^n - 1)}{4^n}$

$= \sum_{n=1}^{\infty} \left(\frac{3^n - 1}{4^n}\right) = \sum_{n=1}^{\infty} \left(\frac{3^n}{4^n} - \frac{1}{4^n}\right) = \sum_{n=1}^{\infty} \left(\left(\frac{3}{4}\right)^n - \left(\frac{1}{4}\right)^n\right)$

$= \underbrace{\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n}_{a} - \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n}_{b}$

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a. $\sum_{n=1}^{\infty} (\frac{3}{4})^n = \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3 + \cdots$

$\therefore a = \frac{3}{4}$, $r = \frac{3}{4} \Rightarrow |r| < 1$ (converges)

$$\text{sum}_{(a)} = \frac{a}{1-r} = \frac{(3/4)}{1-(3/4)} = 3$$

b. $\sum_{n=1}^{\infty} (\frac{1}{4})^n = \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots$

$\therefore a = \frac{1}{4}$, $r = \frac{1}{4} \Rightarrow |r| < 1$ (converges)

$$\text{sum}_{(b)} = \frac{a}{1-r} = \frac{(1/4)}{1-(1/4)} = \frac{1}{3}$$

$\therefore$ The series converges

$$\text{sum} = \text{sum}_{(a)} - \text{sum}_{(b)} = 3 - \frac{1}{3} = \frac{8}{3}$$

Note

Conv. $\mp$ Conv. = Conv.

$$\left.\begin{array}{l}\text{conv.} \mp \text{div.} \\ \text{div} \mp \text{div.}\end{array}\right\} = \text{div.}$$

Example (4)

show that the following series is a geometric series, if so find the convergence?

$$\frac{8}{5} + \frac{6}{5} + \frac{9}{10} + \frac{27}{40} + \frac{81}{160} + \frac{243}{640} + \cdots$$

sol.

The geometric series has the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

$\therefore$ the first term is $a = \frac{8}{5}$

the second term is $ar = \frac{6}{5}$

$\therefore$ the ratio $r = \frac{ar}{a} = \frac{(6/5)}{(8/5)} = \frac{6}{8} = \frac{3}{4}$

$$\therefore \quad ar^2 = \frac{8}{5}\left(\frac{3}{4}\right)^2 = \frac{9}{10}$$

$$ar^3 = \frac{8}{5}\left(\frac{3}{4}\right)^3 = \frac{27}{40}$$

$$ar^4 = \frac{8}{5}\left(\frac{3}{4}\right)^4 = \frac{81}{160}$$

$$ar^5 = \frac{8}{5}\left(\frac{3}{4}\right)^5 = \frac{243}{640}$$

$\therefore$ the series is a geometric series

since $|r| < 1 \rightarrow$ converges

$$\text{sum} = \frac{a}{1-r} = \frac{8/5}{1-(6/5)} = \frac{32}{5}$$

$$\therefore \quad \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{32}{5}$$

Example (5)

Test and find the sum of the following series

$$\sum_{n=1}^{\infty} \frac{5(-1)^n}{4^n} \; ?$$

Sol.

$$\sum_{n=1}^{\infty} \frac{5(-1)^n}{4^n} = \frac{-5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256} - \frac{5}{1024} + \cdots$$

$$\therefore \quad a = \frac{-5}{4}$$

$$ar = \frac{5}{16}$$

$\therefore$ the ratio $r = \frac{ar}{a} = \frac{(5/16)}{(-5/4)} = -\frac{1}{4}$

$\therefore \quad ar^2 = -\frac{5}{4}\left(-\frac{1}{4}\right)^2 = \frac{-5}{64}$

$ar^3 = -\frac{5}{4}\left(-\frac{1}{4}\right)^3 = \frac{5}{256}$

$ar^4 = -\frac{5}{4}\left(-\frac{1}{4}\right)^4 = -\frac{5}{1024}$

$\therefore$ the series is a geometric series with $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$

$|r| < 1 \longrightarrow$ the series converges

$$\text{sum} = \frac{a}{1-r} = \frac{(-5/4)}{1-(-1/4)} = \frac{(-5/4)}{(5/4)} = -1$$

$$\therefore \sum_{n=1}^{\infty} \frac{5(-1)^n}{4^n} = -1$$

P-Series

The form of the series as follows:

$$\sum_{n=1}^{\infty} \frac{K}{n^P} = \frac{K}{1^P} + \frac{K}{2^P} + \frac{K}{3^P} + \frac{K}{4^P} + \cdots$$

the function is fractional (كسري) and the numerator (البسط) is constant K, but the denominator (المقام) is Variable (متغير).

Convergence test

If $P > 1$, then the series converges.
If $P \leq 1$, then the series diverges.

Example (1)

Test the convergence for the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ (b) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$ (d) $\sum_{n=1}^{\infty} e^{-4\ln n}$?

Sol.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

it is a P-series with $P=3$ and $K=1$

$P > 1$, then the series converges

(b) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$

this is a P-series with $K=3$ and $P=1/2$

$P < 1$, diverges.

(c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (22)

this is a p-series with $k=1$ and $p=3/2$

$\because p > 1 \rightarrow$ the series converges

(d) $\sum_{n=1}^{\infty} e^{-4\ln n} = \sum_{n=1}^{\infty} \cancel{e}^{\cancel{\ln} n^{-4}} = \sum_{n=1}^{\infty} n^{-4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

it is a p-series with $k=1$ and $p=4$

$\because p > 1$, then the series converges.

Example (2)

Determine if the infinite series converges or diverges?

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$ (b) $\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^4}$?

sol.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

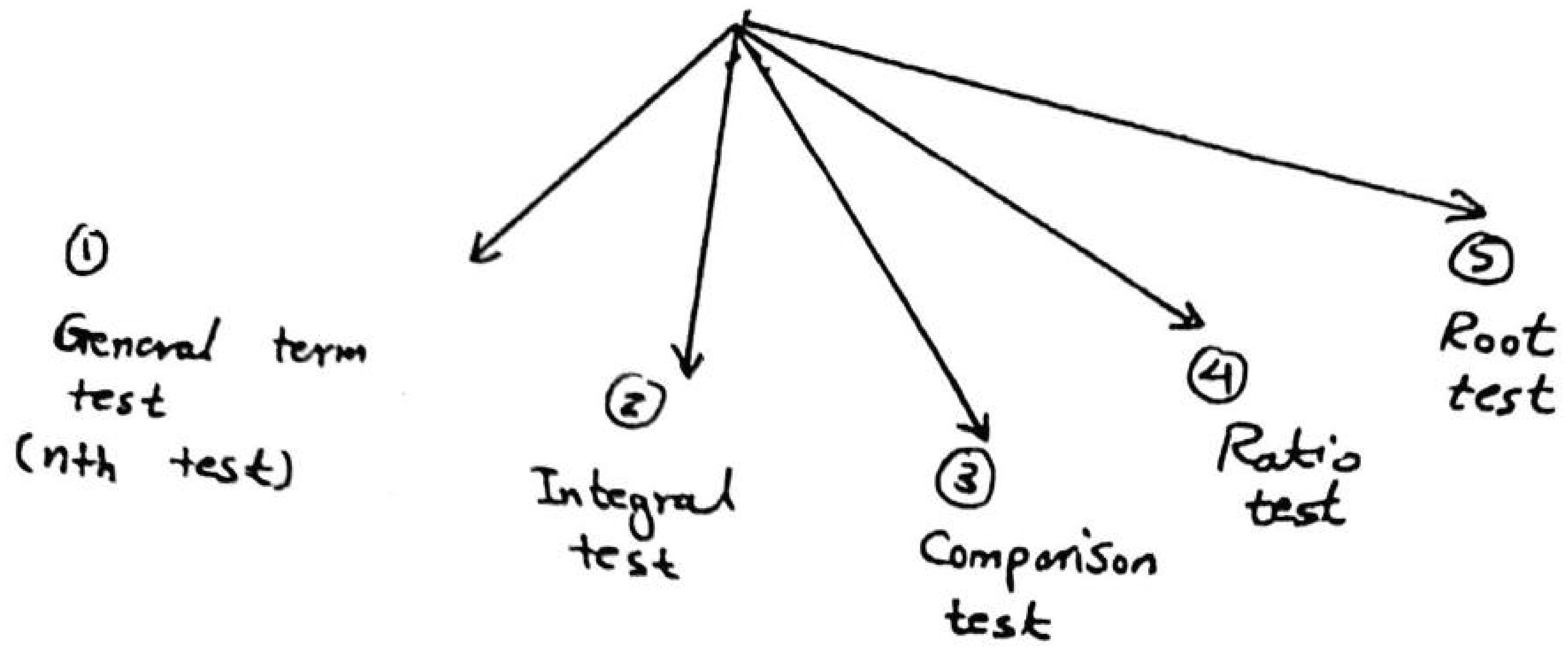
p-series with $p = 4/3 > 1 \rightarrow$ converges.

(b) $\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^4} = 4\sum_{n=1}^{\infty} \frac{n^{1/2}}{n^4} = 4\sum_{n=1}^{\infty} \frac{1}{n^{-1/2} \cdot n^4}$

$= 4\sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$

p-series with $p = 7/2 > 1 \rightarrow$ converges

Convergence TEST



1. General term test

One reason that a series may fail to converge is that its terms don't become small.

If $\sum_{n=1}^{\infty} a_n$ is a series,

(a) If $\lim_{n\to\infty} a_n = 0$ (converges)

(b) If $\lim_{n\to\infty} a_n \neq 0$ (diverges)

Example ①

Test the Convergence for the following using the n-th term:

(1) $\sum_{n=1}^{\infty} \frac{3n}{n-1}$ (2) $\sum_{n=1}^{\infty} \frac{3n^5-4n^3+2n-1}{4n^5+3n^2-n+3}$ ③ $\sum_{n=1}^{\infty} n^2$

(4) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ ⑤ $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ ⑥ $\sum_{n=1}^{\infty} (-1)^{n+1}$?

Sol. (1) $\sum_{n=1}^{\infty} \frac{3n}{n-1}$

$$\lim_{n\to\infty} \frac{3n}{n-1} = \lim_{n\to\infty} \frac{3}{1} \quad \text{(L'-Hopital Rule)}$$

$$= \frac{3}{1} \neq 0 \quad (\therefore \text{ diverges})$$

(2) $\sum_{n=1}^{\infty} \frac{3n^5-4n^3+2n-1}{4n^5+3n^2-n+3}$

$$\lim_{n\to\infty} \frac{3n^5-4n^3+2n-1}{4n^5+3n^2-n+3} = \frac{3}{4} \quad \text{by (L-Hopital Rule)}$$

$$\neq 0 \quad \text{(diverges)}$$

(3) $\sum_{n=1}^{\infty} n^2$

$$\lim_{n\to\infty} n^2 = \infty \quad \text{(diverges)}$$

(4) $\sum_{n=1}^{\infty} \frac{n+1}{n}$

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+1/n}{1} = 1 \neq 0 \quad \text{(diverges)}$$

(5) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

$$\lim_{n\to\infty} \frac{-n}{2n+5} = \lim_{n\to\infty} \frac{-1}{(2+\frac{5}{n})} = -1/2 \neq 0 \quad \text{(diverges)}$$

(6) $\sum_{n=1}^{\infty} (-1)^{n+1}$

$$\lim_{n\to\infty} (-1)^{n+1} = +1, -1, +1, -1, +1 \quad \neq 0 \quad \text{(diverges)}$$

2. Integral test

Let $\{a_n\}$ be a sequence of positive terms.
Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geqslant N$
(where N is a positive integer), then :-

(a) the series $\sum_{n=N}^{\infty} a_n$ converges if $\int_{N}^{\infty} f(x)\,dx$ Converges.

(b) the series $\sum_{n=N}^{\infty} a_n$ diverges if $\int_{N}^{\infty} f(x)\,dx$ diverges.

Example ①

Test the convergence of the following series using the integral test.: $\sum_{n=1}^{\infty} \frac{1}{n}$?

Sol.

$a_n = f(n)$, (where $f(n)$ is a function of x)

$f(x) = \frac{1}{x}$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx = \Big[\ln(x)\Big]_1^{\infty} = \ln(\infty) - \cancel{\ln(1)}^{0} = \infty$$

∴ the series diverges by the integral test.

Example ②

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges or diverges by using the integral test ?

Sol. $a_n = \frac{1}{n^{1.1}} = f(n)$

$\therefore \quad f(x) = \frac{1}{x^{1.1}}$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^{1.1}}\,dx = \int_1^\infty x^{-1.1}\,dx$$

$$= \left[\frac{-10}{x^{1/10}}\right]_1^\infty = \left(\frac{-10}{(\infty)^{1/10}} - \frac{-10}{(1)^{1/10}}\right)$$

$$= 0 - (-10) = \underline{10} \qquad \text{(converges)}$$

Example ③

Test the series (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$, (b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, (c) $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$?

Sol.

(a) $a_n = \frac{1}{n(\ln(n))^2} = f(n)$

$$f(x) = \frac{1}{x(\ln(x))^2}$$

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x(\ln(x))^2}\,dx = \int_2^\infty \frac{(\ln(x))^{-2}}{x}\,dx$$

$$= \left[\frac{(\ln(x))^{-1}}{-1}\right]_2^\infty = \left[\frac{-1}{(\ln(x))^{-1}}\right]_2^\infty$$

$$= \left(\frac{-1}{\ln\infty}\right) - \left(\frac{-1}{\ln 2}\right) = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

(Converges)

(b) $a_n = \frac{1}{n^2} = f(n)$

$$f(x) = \frac{1}{x^2}$$

$$\int_1^\infty f(x) = \int_1^\infty \frac{1}{x^2}\,dx = \int_1^\infty x^{-2}\,dx = \left[\frac{-1}{x}\right]_1^\infty = \left(\frac{-1}{\infty}\right) - \left(\frac{-1}{1}\right)$$

$$= 0 + 1 = 1 \qquad \text{(converges)}$$

(c) $a_n = \frac{1}{\sqrt{n}} = f(n)$

$$\therefore\ f(x) = \frac{1}{\sqrt{x}}$$

$$\int_3^\infty f(x)\,dx = \int_3^\infty \frac{1}{\sqrt{x}}\,dx = \int_3^\infty x^{-\frac{1}{2}}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_3^\infty$$

$$= \left[2\sqrt{x}\right]_3^\infty = 2\left[\sqrt{\infty} - \sqrt{3}\right] = \infty$$

(diverges)

3. Comparison test

Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms.

(a) If $a_n \leqslant b_n$, if $\sum b_n$ converges, then $\sum a_n$ also converges.

(b) If $a_n \geqslant b_n$, if $\sum b_n$ diverges, then $\sum a_n$ diverges.

NOTE: we have to select the series $\sum b_n$ whose convergence is known (geometric series, P-series ....etc)

* which means that to apply the Comparison test we need to recognise another series diverges or converges by another test.

Example 1

Test for Convergence the following series $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ using the comparison test?

Sol. $$\sum_{n=1}^{\infty} \frac{1}{n^2+3} = \sum a_n$$

To apply the comparison test, we have to use another series whose convergence is known.

$$a_n = \frac{1}{n^2+3}$$

$$b_n = \frac{1}{n^2}$$

$$\therefore (n^2+3) > n^2$$

$$\therefore \frac{1}{n^2+3} < \frac{1}{n^2} \Rightarrow \therefore \boxed{a_n < b_n}$$

the series $\sum b_n = \sum \frac{1}{n^2}$ converges using the p-series with $p = 2 > 1$.

$\therefore$ The series $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+3}$ also converges using the comparison test. (part a)

Example 2

Which one of the following series converge, and which divage?

(a) $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ (b) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ (c) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ (d) $\sum_{n=1}^{\infty} \frac{n^3}{e^{n^4}+17}$?

Sol.

(a) $a_n = \frac{1}{\sqrt{n-2}}$

we select $b_n = \frac{1}{\sqrt{n}}$, the series $\sum_{n=3}^{\infty} b_n = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$

to test the series $\sum b_n$ using p-series

$$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=3}^{\infty} \frac{1}{n^{1/2}} \quad \text{(p-series)}$$

$\therefore p = 1/2 < 1$ (diverges)

$$\sqrt{n-2} < \sqrt{n}$$

$$\therefore a_n > b_n$$

$\therefore$ the series $\sum a_n = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ diverges too using the comparison test (part b).

(b) $a_n = \frac{1}{\ln(n)}$

we select $b_n = \frac{1}{n}$ to get the series $\sum b_n = \sum \frac{1}{n}$ (p-series)

$\frac{1}{\ln(n)} > \frac{1}{n}$ (because $\ln(n) < n$)

$\therefore a_n > b_n$

the series $\sum b_n = \sum \frac{1}{n}$ with $p=1$ (diverges)

$\therefore$ the series $\sum a_n = \sum \frac{1}{\ln(n)}$ diverges too using the comparison test (part b).

(c) $a_n = \frac{\sin^2 n}{n^3}$

we select $b_n = \frac{1}{n^3}$ to get the p-series $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$

this p-series $\sum b_n$ converges because $p = 3 > 1$.

$\frac{\sin^2 n}{n^3} \leqslant \frac{1}{n^3}$ (because $\sin^2 n \leq 1$)

$\therefore$ $a_n \leqslant b_n$

The series $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ with $p = 3 > 1$ converges.

$\therefore$ The series $\sum a_n = \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ also converges using the comparison test (part a).

(d) $a_n = \frac{n^3}{e^{n^4} + 17}$

we select $b_n = \frac{n^3}{e^{n^4}}$

$\frac{n^3}{e^{n^4} + 17} < \frac{n^3}{e^{n^4}}$, (because $(e^{n^4} + 17) > e^{n^4}$)

$\therefore$ $a_n < b_n$

we test the series $\sum b_n = \sum_{n=1}^{\infty} \frac{n^3}{e^{n^4}}$ by integral test because $\sum b_n$ is neither a geometric series nor a p-series.

$\therefore$ $b_n = \frac{n^3}{e^{n^4}} = f(n)$

$\therefore$ $f(x) = \frac{x^3}{e^{x^4}}$

$$\therefore \int_1^{\infty} \frac{x^3}{e^{x^4}}\,dx = \int_1^{\infty} x^3 \cdot e^{-x^4}\,dx = -\frac{1}{4}\left[e^{-x^4}\right]_1^{\infty}$$

$$= -\frac{1}{4}\left[\frac{1}{e^{\infty}} - \frac{1}{e^1}\right] = -\frac{1}{4}\left[0 - \frac{1}{e^1}\right] = \frac{1}{4 \cdot e^1}$$

$\therefore$ the series $\sum b_n$ converges by integral test.

$\therefore$ the series $\sum a_n$ also converges by Comparison test (part a).

The Limit Comparison Test

This test is particularly useful for series in which a_n is a rational function of n.

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer) [a_n and b_n have positive terms].

(a) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(c) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

NOTE: Helpful if the conditions of the direct comparison test are not satisfied. تتحقق
($a_n \leq b_n$, $\sum a_n$ converges if $\sum b_n$ converges)
($a_n \geq b_n$, $\sum a_n$ diverges if $\sum b_n$ diverges.)

مثل like $b_n > a_n$ and $\sum b_n$ diverges
$b_n < a_n$ and $\sum b_n$ converges.

Ratio Test

let $\sum u_n$ is the infinite Series of Postive terms

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = P$$

* The Series $\sum u_n$ be Converge when $P < 1$
* The Series $\sum u_n$ be diverges when $P > 1$
* This test imposible when $P = 1$

Example: Determine if the two Series below Converge or diverge?

1) $u_n = \frac{n!}{3^n}$, $U_{n+1} = \frac{(n+1)!}{3^{n+1}}$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{1}{3}(n+1$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{3} \lim_{n\to\infty} (n+1) = \infty \Rightarrow \text{This series is divergence}$$

$$u_n = \frac{n+1}{n} \cdot \frac{1}{4^{n-1}}$$

Solution: $u_{n+1} = \frac{n+2}{n+1} \cdot \frac{1}{4^n}$

$$\frac{u_{n+1}}{u_n} = \frac{n+2}{n+1} \cdot \frac{1}{4^n} \cdot \frac{n \cdot 4^{n-1}}{n+1}$$

$$= \frac{n^2+2n}{n^2+2n+1} \cdot \frac{1}{4}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{4} \lim_{n\to\infty} \frac{2n+2}{2n+2} = \frac{1}{4}$$

So this Series is convergent

Root Test

let $\sum u_n$ is the infinite Series

$$\lim_{n\to\infty} \sqrt[n]{u_n} = P$$

* $\sum u_n$ converge if $P<1$
* $\sum u_n$ diverge if $P>1$
* if $P=1$ the Series maybe converge or diverge

Example: Determine if this Series converge or diverge

① $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Solution: $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$\because P<1$

$\therefore$ this Series is converge.

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$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

Solution: $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{n^{2/n}} = \frac{2}{(n^{\frac{1}{n}})^2}$

$$\lim_{n \to \infty} \frac{2}{(n^{1/n})^2} = \frac{2}{1^2} = 2$$

$\because P > 1$

$\therefore$ the series is divergence.

Infinite Sequences and Infinite Series

1. Infinite Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, a_4, \ldots\ldots, a_n$$

تمثل

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the TERMS of the sequence.

For example, the sequence

$$2, 4, 6, 8, 10, 12, \ldots\ldots, 2n$$

has first term $a_1 = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the index of a_n.

Note

Order is important. The sequence $2, 4, 6, 8, \ldots\ldots$ is not the same as the sequence $4, 2, 6, 8 \ldots\ldots$.

Definition: the infinite sequence of numbers is a function whose domain is the set of positive integers.

العام السلوك

for example, the general behaviour of the following sequences are:

(1) $a_n = 2n = 2, 4, 6, 8, 10, 12, \ldots\ldots$

(2) $a_n = \frac{1}{n} = 1, 1/2, 1/3, 1/4, 1/5, 1/6 \cdots$

(3) $a_n = \frac{n+1}{n} = 2, 3/2, 4/3, 5/4, 6/5 \cdots\cdots$

So, $\frac{n+1}{n}$ is also called the general term of the sequence.

Example

write the first five terms of the following sequences:

1. $\left[\frac{2n-1}{3n+2}\right]$ or $a_n = \frac{2n-1}{3n+2}$ or $\left\{\frac{2n-1}{3n+2}\right\}$

$n = 1, 2, 3, 4, 5$ "five terms"

$a_1 = 1/5$, $a_2 = 3/8$, $a_3 = 5/11$, $a_4 = 7/14$, $a_5 = 9/17$

$\therefore$ The five terms are $[1/5, 3/8, 5/11, 7/14, 9/17]$

2. $\left[\frac{1+(-1)^n}{n}\right]$ or $a_n = \frac{1+(-1)^n}{n}$

$n = 1, 2, 3, 4, 5$

$a_1 = 0$, $a_2 = 1$, $a_3 = 0$, $a_4 = 1/2$, $a_5 = 0$

$\therefore$ The first five terms are $[0, 1, 0, 1/2, 0]$

التقارب التباعد
Convergence and Divergence

تقترب
Sometimes the numbers in a sequence approach a single value as the index "n" increases.

For example

(1) $\left[1, 1/2, 1/3, 1/4, 1/5, \ldots\ldots, 1/n, \ldots\right]$
0.5

the terms approach zero as n gets large.

(2) $\left\{0, 1/2, 2/3, 3/4, 4/5, \ldots\ldots, 1-\frac{1}{n}, \ldots\right\}$

the terms approach 1.

تقترب
In this case, it can be said that the sequence converges to this specific number.

Definition

If a_n converges L, we write

$\boxed{\lim_{n\to\infty} a_n = L}$, or simply $a_n \xrightarrow[\text{to}]{\text{converge}} L$ and call L

حد
the limit of the sequence.

بالمقابل ابداً لا
On the other hand, if the sequence never converging to a single value or whose terms get larger than any number as n increases, then it can be said that the sequence

تتباعد
diverges to infinity or negative infinity.

for example,

(1) $\{1, -1, 1, -1, 1, -1, \ldots\ldots, (-1)^{n+1}, \ldots\}$

(2) $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ldots\ldots, \sqrt{n}, \ldots\}$.

Calculating Limits of Sequences

Theorm 1:

let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules (القواعد) hold

if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. (Converges)

1. Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = A + B$

2. Difference (الفرق) Rule: $\lim_{n\to\infty} (a_n - b_n) = A - B$

3. Constant Multiple Rule: $\lim_{n\to\infty} (K \cdot b_n) = K \cdot B$

   (K is any constant number)

4. Product (حاصل الضرب) Rule: $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$

5. Quotient (القسمة) Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

Notes

$\frac{\infty}{\text{كمية}} = \infty$, $\frac{\text{كمية}}{\infty} = 0$, $\infty \mp \infty = \infty$, $\infty \mp \text{كمية} = \infty$

Examples

a. $\lim_{n\to\infty} (-\frac{1}{n}) = -1 \cdot \lim_{n\to\infty} (\frac{1}{n}) = -1 \cdot \frac{1}{\infty} = -1 \cdot 0 = 0$

(constant multiple Rule)

b. $\lim_{n\to\infty} (\frac{n-1}{n}) = \lim_{n\to\infty} (1 - \frac{1}{n})$

$= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$

(difference Rule)

c. $\lim_{n\to\infty} \frac{5}{n^2} = 5 \cdot \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 5 \cdot 0 \cdot 0 = 0$

product
(difference Rule)

d. $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{(4/n^6)-7}{1+(3/n^6)} = \frac{0-7}{1+0} = -7$

"sum + quotient Rules"

Theorm 2:

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0$ $(|x|<1)$ $-1<x<1$

$= \infty$ $(|x| \geq 1)$ $x>1$, $x<-1$

5. $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ (any x)

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

7. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$ (L'Hopitals Rule)

Examples

These are examples of the limits:

(a) $\left[\frac{\ln(n^2)}{n}\right]$

$$\lim_{n\to\infty} \frac{\ln(n^2)}{n} = \lim_{n\to\infty} \frac{2\ln(n)}{n} = 2 \cdot \lim_{n\to\infty} \frac{\ln(n)}{n} = 2 \cdot 0 = 0$$

formula 1 (converges)

(b) $\left[\sqrt[n]{n^2}\right]$

$$\lim_{n\to\infty} \left[\sqrt[n]{n^2}\right] = \lim_{n\to\infty} (n^2)^{1/n} = \lim_{n\to\infty} (n)^{2 \cdot 1/n}$$

$$= \lim_{n\to\infty} (n^{1/n})^2 = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2$$

$$= (1)^2$$

formula 2 (converges)

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(c) $\left[\sqrt[n]{3n}\right]$

$$\lim_{n\to\infty} \left(\sqrt[n]{3n}\right) = \lim_{n\to\infty} 3^{1/n} \cdot \left(n^{1/n}\right) = 1 \cdot 1 = 1$$

formula 3 with $x=3$ and formula 2
(converges)

(d) $\left[\left(-\frac{1}{2}\right)^n\right]$

$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 0$ formula 4 with $|x|<1$
(converges)

(e) $\left[2^n\right]$

$\lim_{n\to\infty} (2)^n = \infty$ formula 4 with $|x|>1$ (diverges)

Note

أذا كانت a_n كسرية

اذا كانت درجة البسط = درجة المقام فان

$\lim_{n\to\infty} a_n = \dfrac{\text{معامل اكبر أس بالبسط}}{\text{معامل اكبر أس بالمقام}}$

أذا كانت درجة البسط اكبر من درجة المقام فان

$\lim_{n\to\infty} a_n = \infty$

اذا كانت درجة البسط اقل من درجة المقام فان

$\lim_{n\to\infty} a_n = 0$

سيتم ايجاد قيمة $\lim_{n\to\infty} a_n$ أما بطريقة القسمة على أكبر أس أو بقاعدة (L'Hopital) .

Examples

(a) $\left[\dfrac{3n^2-5n}{5n^2+2n-6}\right]$

the order of the numerator is the same as the order of the denominator. the result is 3/5

$$\lim_{n\to\infty}\frac{3n^2-5n}{5n^2+2n-6} = \lim_{n\to\infty}\frac{3-5/n}{5+2/n-6/n^2} = \frac{3-0}{5+0-0}$$

$$= \frac{3}{5} \quad \text{(Converges)}$$

(b) $\left[\left(\dfrac{2n+3}{3n-7}\right)^4\right]$

$$\lim_{n\to\infty}\left(\frac{2n+3}{3n-7}\right)^4 = \lim_{n\to\infty}\left(\frac{2+3/n}{3-7/n}\right)^4 = \left(\frac{2+0}{3-0}\right)^4 = \left(\frac{2}{3}\right)^4$$

(Converges)

(c) $\left[\dfrac{n^3+1}{n^2+3}\right]$

the order of the numerator > the order of the denominator

∴ the resut is ∞

$$\lim_{n\to\infty}\left(\frac{n^3+1}{n^2+3}\right) = \lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\bar{f}(n)}{\bar{g}(n)} \quad \text{(L'Hôpital Rule)}$$

$$= \lim_{n\to\infty}\frac{3n^2}{2n} = \lim_{n\to\infty}\frac{6n}{2} = \infty$$

(diverges)

(e) $\left\{\frac{n}{n^2+3}\right\}$

the order of numerator < the order of denominator

$\therefore$ the results is zero.

$$\lim_{n\to\infty}\left(\frac{n}{n^2+3}\right) = \lim_{n\to\infty}\frac{1}{2n+0} = \frac{1}{\infty} = 0$$

(converges)

Differential Equation

ordinary (derivative)

ex/ $\frac{dy}{dx} = x + 10$

dependent   independents

Partial

ex/ $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 5z$

independent   dependent

The order & The degree

The order → highest order derivative

The degree → highest power to highest order derivative

ex/ $\frac{dy}{dx} + \cos x = 0 \longrightarrow$ first order, first degree

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5xy = 0 \longrightarrow$ second order, first degree

$\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} = xy \longrightarrow$ second order, third degree

$(y''')^2 + 2(y'')^2 + y = xy \longrightarrow$ third order, second degree

Types of solution

General solution

Particular solution

Singular solution

ex/ $y = \sin(x + c)$

$y = \sin(x+1)$

$y = 1$

ex/ show that $y = \sin(x+c)$, $y = \sin(x+1)$ & $y = 1$ are solution for the D.E $(y')^2 + y^2 = 1$

① $y = \sin(x+c) \rightarrow y' = \cos(x+c)$

$\therefore (\cos(x+c))^2 + (\sin(x+c))^2 = 1 \longrightarrow$ general solution because the c constant

② $y = \sin(x+1) \rightarrow y' = \cos(x+1)$

$\therefore (\cos(x+1))^2 + (\sin(x+1))^2 = 1 \longrightarrow$ Particular solution

③ $y = 1 \longrightarrow y' = 0$

$\therefore (0)^2 + (1)^2 = 1 \longrightarrow$ singular solution ex/ Show that $y = 3e^{2x} - e^{-2x}$ is solution for $\grave{y} - 4y = 0$?

$\grave{y} = 6e^{2x} + 2e^{-2x}$

$\ddot{y} = 12e^{2x} - 4e^{-2x}$

$\therefore 12e^{2x} - 4e^{-2x} - 4(3e^{2x} - e^{-2x}) = 0$

$\cancel{12e^{2x}} - \cancel{4e^{-2x}} - \cancel{12e^{2x}} - \cancel{4e^{-2x}} = 0$ Singular solution

$\therefore$ y is solution of D.E

ex/ Show that $y = \sin 2x$ is a solution for D.E $\ddot{y} + 4y = 0$?

$\bar{y} = 2\cos 2x$

$\ddot{y} = -4\sin 2x$

$\therefore -4\sin 2x + 4(\sin 2x) = 0$

$-4\cancel{\sin 2x} + 4\cancel{\sin 2x} = 0$ $\therefore$ $y = f(x)$ is the solution of D.E

Note:-

$\sin x = \cos x$

$\cos x = -\sin x$

$\tan x = \sec^2 x$

$\csc x = -\csc x \cot x$

$\sec x = \sec x \tan x$

$\cot x = -\csc^2 x$

D.E

separable — فصل | homogeneous — المتجانس | exact — بالضبط | linear — خطي | Bernoulie

① separable

$$f(x)\,dx + g(y)\,dy = 0 \quad \underline{\underline{or}} \quad M\,dx + N\,dy = 0$$

where $M = f(x)$ or constant

$N = f(y)$ or constant

شكل المعادلة نهائي $\int f(x)\,dx + \int g(y)\,dy = c$

ex/ solve D.E $(x+1)\dfrac{dy}{dx} = y$?

sol/ $(x+1)\dfrac{dy}{dx} = y$

نقل ① $\dfrac{dy}{dx} = \dfrac{y}{(x+1)}$

نفصل ② $\dfrac{dy}{y} = \dfrac{dx}{(x+1)}$

نكامل ③ $\int \dfrac{dy}{y} = \int \dfrac{dx}{(x+1)}$

$\therefore \ln(y) = \ln(x+1) + c$

or $e^{\ln y} = e^{\ln(x+1) + c}$ $\xrightarrow{④}$ $\therefore y = A\,(x+1)$

$A = e^{c}$

ex / solve the D.E $x(2y-3)\,dx + (x^2+1)\,dy = 0$?

$$\frac{x}{(x^2+1)}dx + \frac{1}{(2y-3)}dy = 0$$

$$\int \frac{x}{(x^2+1)}dx + \int \frac{1}{(2y-3)}dy = 0$$

$$= \frac{1}{2}\int 2x(x^2+1)^{-1}dx + \frac{1}{2}\int 2(2y-3)^{-1}dy$$

$$\therefore \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\ln(2y-3) + c = 0$$

ex / solve the D.E $\frac{dy}{dx} = \frac{x\sqrt{1+y^2}}{2-3x^2}$?

$$\frac{dy}{\sqrt{1+y^2}} = \frac{x\,dx}{(2-3x^2)}$$

$$\therefore \int \frac{dy}{\sqrt{1+y^2}} - \int \frac{x}{(2-3x^2)}dx = 0$$

$$= \sinh^{-1}(y) + \frac{1}{6}\ln(2-3x^2) + c = 0$$

② homogeneous D.E

$$f(x,y)\,dx + g(x,y)\,dy = 0$$

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

$$g(\lambda x, \lambda y) = \lambda^n g(x,y)$$

$$\therefore \frac{f(\lambda x, \lambda y)}{g(\lambda x, \lambda y)} = \frac{f(x,y)}{g(x,y)} \longrightarrow \therefore \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = F\left(\frac{y}{x}\right) \quad ①$$

To solve 1 Put $v = \frac{y}{x} \longrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ..... ②

$$v + x\frac{dv}{dx} = F(v)$$

$F(v) - v = x\frac{dv}{dx}$ $\therefore$ by sparble

$\frac{dv}{F(v)-v} = \frac{dx}{x}$ by integration both side

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x} \longrightarrow \ln x + c$$

ex ① Find the general of the following D.E

$(x^3+y^3)\,dx - 3xy^2\,dy = 0$

sol/ $\frac{dy}{dx} = \frac{(x^3+y^3)}{3xy^2}$

$\frac{dy}{dx} = \frac{(\lambda^3 x^3 + \lambda^3 y^3)}{3\lambda x(\lambda y)^2} = \frac{\lambda^3 (x^3+y^3)}{\lambda^3\, 3xy^2} = \frac{(x^3+y^3)}{3xy^2}$

$\therefore \frac{f(\lambda x, \lambda y)}{g(\lambda x, \lambda y)} = \frac{f(x,y)}{g(x,y)}$ the D.E is homgenus

$\frac{dy}{dx} = \frac{x^3+y^3}{3xy^2}$ dividing by x^3

$\frac{dy}{dx} = \frac{1+(\frac{y}{x})^3}{3(\frac{y}{x})^2}$

assume $v = \frac{y}{x} \rightarrow y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{1+v^3}{3v^2}$ (spearable D.E)

$x\frac{dv}{dx} = \frac{1-2v^3}{3v^2}$

where $C_1 = -2c$

$\therefore \int \frac{3v^2}{1-2v^3}\,dv \rightarrow \int \frac{dx}{x}$

$-\frac{1}{2}\ln(1-2v^3) = \ln(x) + c \rightarrow \ln\left(1-2\left(\frac{y}{x}\right)^3\right) + \ln x = C_1$

ex② / find the general solution $(x e^{\frac{y}{x}} + y)dx - x dy = 0$

$$\frac{dy}{dx} = \frac{(x e^{\frac{y}{x}} + y)}{x} \longrightarrow \frac{dy}{dx} = \frac{(\lambda x e^{\frac{\lambda y}{\lambda x}} + \lambda y)}{\lambda x}$$

$$= \frac{\cancel{\lambda}(x e^{\frac{y}{x}} + y)}{\cancel{\lambda} x} \longrightarrow \frac{(x e^{\frac{y}{x}} + y)}{x}$$

$\therefore$ D.E is homogeneous dividing by (x)

$$\frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x}$$

let $v = \frac{y}{x} \longrightarrow y = vx \longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore v = x\frac{dv}{dx} = e^{v} + v$$

$$x\frac{dv}{dx} = e^{v} + v - v$$

$$\int \frac{dv}{e^{v}} = \int \frac{dx}{\cancel{x}}$$

$$-e^{-v} = \ln(x) + c$$

$$\therefore \ln(x) + e^{\left(-\frac{y}{x}\right)} = c$$

3 Exact D.E :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M(x,y) =$ معلومة $\longrightarrow \dfrac{\partial M}{\partial y} =$ مشتقة

$N(x,y) =$ معلومة $\longrightarrow \dfrac{\partial N}{\partial x} =$ مشتقة

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ Exact

$M = \dfrac{\partial F}{\partial x}$ & $N = \dfrac{\partial F}{\partial y}$

$$F(x,y) = \int M(x,y) \cdot \partial x + c_y$$

or $F(x,y) = \int N(x,y) \cdot \partial y + C_x$

$\dfrac{\partial f(x,y)}{\partial y} = N$ to get $\dfrac{\partial c_y}{\partial y}$

or $\dfrac{\partial F(x,y)}{\partial x} = M$ to get $\dfrac{\partial C_x}{\partial x}$

ex/ solve the D.E $(x+y)dx+(x+y^2)dy=0$

Sol/ $M(x,y) = x+y \longrightarrow \frac{\partial M}{\partial y} = 1$

$N(x,y) = x+y^2 \longrightarrow \frac{\partial N}{\partial x} = 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\quad \therefore$ D.E is exact

$M = \frac{\partial F}{\partial x} \longrightarrow \partial F = M \cdot \partial x$

$\int \partial F = \int M \cdot \partial x \longrightarrow \int \partial F = \int (x+y)\, \partial x$

$F = \frac{x^2}{2} + xy + C_y$ --- ①

بقسمة المعادلة ① على (∂y) فتصبح

$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{2}\right) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y} C_y$

$\because \frac{\partial F}{\partial y} = N$

$\therefore N = 0 + x + \frac{\partial}{\partial y} C_y$

$\cancel{x} + y^2 = \cancel{x} + \frac{\partial}{\partial y} C_y \longrightarrow \partial C_y = y^2\, \partial y$

$\int \partial C_y = \int y^2\, \partial y \longrightarrow \therefore C_y = \frac{y^3}{3} + C$ نعوضها في ①

$\therefore F = \frac{x^2}{2} + xy + \frac{y^3}{3} + C$ ← The general solution

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ex/ solve the D.E $(x^3-5x^4y^3)dx-(3x^5y^2-\sin y)dy = 0$

sol/ $(x^3-5x^4y^3)dx+(-3x^5y^2+\sin y)dy=0$

$M(x,y)=x^3-5x^4y^3 \longrightarrow \frac{\partial M}{\partial y} = -15x^4y^2$

$N(x,y) = -3x^5y^2+\sin y \longrightarrow \frac{\partial N}{\partial x} = -15x^4y^2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ D.E is exact

$\because M = \frac{\partial F}{\partial x} \longrightarrow \partial f = M \cdot \partial x \longrightarrow \int \partial F = \int M \cdot \partial x$

$F = \int (x^3-5x^4y^3)\,\partial x$

$F = \frac{x^4}{4} - \frac{\cancel{5}}{\cancel{5}}x^5y^3 + c_y$ ---①

$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^4}{4}\right) - \frac{\partial}{\partial y}(x^5y^3) + \frac{\partial c_y}{\partial y}$

$\because \frac{\partial F}{\partial y} = N \quad \therefore N = 0 - 3y^2x^5 + \frac{\partial c_y}{\partial y}$

$\cancel{-3x^5y^2} + \sin y = \cancel{-3y^2x^5} + \frac{\partial c_y}{\partial y}$

$\partial c_y = \sin y \cdot \partial y \longrightarrow \int \partial c_y = \int \sin y \cdot dy$

$\therefore c_y = \cos y + c$ نعوض ①

$\therefore F = \frac{x^4}{4} - \frac{\cancel{5}x^5}{\cancel{5}}y^3 + \cos y + c$ The general solution ex/ $(3e^{3x}\cdot y-2x)\,dx+(e^{3x}+1)\,dy=0$

sol/

$$M(x,y)=3e^{3x}\cdot y-2x \longrightarrow \frac{\partial M}{\partial y}=3e^{3x}$$

$$N(x,y)=e^{3x}+1 \longrightarrow \frac{\partial N}{\partial x}=3e^{3x}$$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ D.E is exact

$$M=\frac{\partial F}{\partial x} \longrightarrow \partial F=M\cdot\partial x \longrightarrow \int\partial F=\int M\cdot\partial x$$

$$F=\int 3e^{3x}\cdot y-2x\;\partial x$$

$$\therefore F=e^{3x}\cdot y-\frac{\cancel{2}x^2}{\cancel{2}}+c_y \quad ---①$$

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}e^{3x}\cdot y-\frac{\partial}{\partial y}x^2+\frac{\partial c_y}{\partial y}$$

$\downarrow$

$$N=e^{3x}-0+\frac{\partial c_y}{\partial y}$$

$$\cancel{e^{3x}}+1=\cancel{e^{3x}}+\frac{\partial c_y}{\partial y} \longrightarrow \therefore \partial c_y=\partial y$$

$\int\partial c_y=\int\partial y \longrightarrow \therefore c_y=y+c$ نعوضها في ①

$\therefore F=e^{3x}\cdot y-x^2+y+c$ The general solution

④ linear D.E :-

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

$P(x) =$ معامل & $Q(x) =$ قيمة

$$IF = e^{\int P(x)\,dx}$$ (IF) إيجاد

Integrating factor

$$IF \cdot y = \int IF \cdot Q(x)\,dx + c$$

ex/ solve the D.E $x\frac{dy}{dx} = x^2 + 3y$

sol/ $\frac{dy}{dx} = \frac{x^2+3y}{x} \longrightarrow \frac{dy}{dx} = \frac{x^{\cancel{2}}}{\cancel{x}} + \frac{3y}{x}$

$$\frac{dy}{dx} - \frac{3}{x}y = x$$

$\therefore P(x) = -\frac{3}{x}$ & $Q(x) = x$

$$IF = e^{\int P(x)\,dx} \longrightarrow e^{\int -\frac{3}{x}\,dx} \longrightarrow e^{-3\ln x} \longrightarrow e^{\ln x^{-3}} \longrightarrow \frac{1}{x^3}$$

$$\therefore IF \cdot y = \int IF \cdot Q(x)\,dx + c$$

$$\frac{1}{x^3} \cdot y = \int \frac{1}{x^{\cancel{3}}} \cdot \cancel{x}\,dx + c \longrightarrow \int \frac{x^{-2+1}}{-2+1}dx + c$$

$\frac{y}{x^3} = -\frac{1}{x} + c$ The general solution ex/ solve the D-E $\frac{dy}{dx} = \frac{3y - 4x^5}{x}$

sol/ $\frac{dy}{dx} + \left(\frac{-3}{x}\right) y = -4x^4$

$\therefore P(x) = -\frac{3}{x}$ & $Q(x) = -4x^4$

$IF = e^{\int P(x)\,dx} \longrightarrow e^{\int -\frac{3}{x}\,dx} \longrightarrow e^{-3\ln x} \longrightarrow e^{\ln x^{-3}} \longrightarrow \frac{1}{x^3}$

$\therefore IF \cdot y = \int IF \cdot Q(x)\,dx + c$

$\frac{1}{x^3} \cdot y = \int \frac{1}{\cancel{x^3}} (-4\cancel{x^4})\,dx + c$

$\frac{y}{x^3} = \int -4x\,dx + c \longrightarrow \frac{y}{x^3} = \frac{-\cancel{4}x^2}{\cancel{2}} + c$

ضرب المعادلة

$\therefore \frac{y}{x^3} = -2x^2 + c \Big\} * x^3 \longrightarrow \therefore y = -2x^5 + cx^3$

The general solution

ex/solve $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$

sol/ $P(x) = \cot x$ & $Q(x) = 5e^{\cos x}$

$IF = e^{\int \cot x\,dx} \longrightarrow e^{\ln \sin x} \longrightarrow \sin x$

$IF \cdot y = \int IF \cdot Q(x)\,dx + c$

$y \cdot \sin x = \int \sin x \cdot 5e^{\cos x}\,dx + c$

$y \cdot \sin x = -5e^{\cos x} + c$ the general solution

Note

$\tan = \frac{\sin}{\cos}$

$\cot = \frac{\cos}{\sin}$

$\frac{1}{\sin} = \ln \sin$

⑤ Bernollies D.E :-

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n$$

اقسم المعادلة على y^n اكبر power مرفوع لل (y)

$$\left[\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n\right\} \div y^n$$

$$\therefore y^{-n} \frac{dy}{dx} + P(x)\, y^{1-n} = Q(x)$$

$$z = y^{1-n} \longrightarrow \frac{dz}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

Integrating by part

$$\int u \cdot dv = u \cdot v - \int v \, du$$

ex/ $\int e^{4x} \cdot (-4x) \, dx$

$u = -4x \longrightarrow du = -4dx$

$dv = e^{4x} dx \longrightarrow v = \frac{1}{4} e^{4x}$

$$\int u\,dv = -\cancel{4}x \cdot \frac{1}{\cancel{4}} e^{4x} - \int \frac{1}{4} e^{4x} \cdot (-4) dx$$

$$= -x e^{4x} +$$

ex/ solve the D.E $\frac{dy}{dx} - y = x y^5$

$$\frac{dy}{dx} - y = x y^5 \quad \Big\} \div y^5$$

$$\boxed{y^{-5}\frac{dy}{dx} - y^{-4} = x} \longrightarrow ① \qquad \because\ y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)$$

$$\boxed{z = y^{-4}} \longrightarrow ②$$

$$\frac{dz}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}$$

$$= -4\,y^{-5}\,\frac{dy}{dx} \longrightarrow ③$$

اعوض 2 & 3 في ①

$$\boxed{-\frac{1}{4}\cdot\frac{dz}{dx} = y^{-5}\frac{dy}{dx}}$$

$$\therefore\ -\frac{1}{4}\frac{dz}{dx} - z = x$$

اضرب المعادلة (-4)

$$\frac{dz}{dx} + 4z = -4x \qquad \text{by linear}$$

$$P(x) = 4 \ \& \ Q(x) = -4x$$

$$IF = e^{\int 4\,dx} \longrightarrow e^{4x}$$

$$IF\cdot z = \int IF\cdot Q(x)\,dx + c$$

$$e^{4x}\cdot z = \int e^{4x}\cdot(-4x)\,dx + c \longrightarrow \text{Integrating by Part}$$

$$e^{4x} \div \left[e^{4x}\cdot y^{-4} = -x e^{4x} + \frac{1}{4}e^{4x} + c\right] \qquad \int u\cdot dv = vu - \int v\cdot du$$

$$= -x e^{4x} + \frac{1}{4}e^{4x} + c$$

$$\therefore\ \frac{1}{y^4} = -x + \frac{1}{4} + c\,e^{-4x}$$

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ex/ solve $\frac{dy}{dx} + \frac{1}{x} y = \ln x \cdot y^2$?

Sol/ $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} y^{-1} = \ln x \quad —①$

$z = y^{-1} \quad —②$ $\qquad \frac{dz}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$

$\frac{dz}{dx} = -y^{-2} \frac{dy}{dx} \quad —③$ $\qquad$ نعوض 3 & 2 في ①

$-\frac{dz}{dx} + \frac{1}{x} z = \ln x$] solve by linear mothed

نضرب بالسالب (-1)

$\frac{dz}{dx} - \frac{1}{x} z = -\ln x$

$\therefore P(x) = \frac{-1}{x}$ & $Q(x) = -\ln x$

$IF = e^{-\int \frac{1}{x} dx} \longrightarrow e^{-\ln x} \longrightarrow e^{\ln x^{-1}} \longrightarrow \therefore IF = \frac{1}{x}$

$IF \cdot z = \int IF \cdot Q(x)\, dx + c$

$\frac{1}{x} \cdot z = \int \frac{1}{x} \cdot (-\ln x)\, dx + c$

$\frac{1}{x} z = -\frac{(\ln x)^2}{2} + c$ } نضرب الطرفين في (x)

$\therefore \frac{1}{y} = \frac{-x(\ln x)^2}{2} + c \cdot x$ $\qquad$ The general solution

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ex/ solve the D.E $y(6y^2-x-1)\,dx+2x\,dy=0$

sol/ $y(6y^2-x-1)\,dx+2x\,dy=0 \;\}\div dx$

$$2x\frac{dy}{dx}+y(6y^2-x-1)=0$$

$$2x\frac{dy}{dx}+6y^3-\boxed{xy-y}=0$$

$$2x\frac{dy}{dx}-(x+1)\cdot y=-6y^3 \quad \}\div 2x$$

$$\frac{dy}{dx}-\frac{(x+1)}{2x}y=-\frac{3}{x}y^3 \quad \}\div y^3$$

$$\frac{1}{y^3}\frac{dy}{dx}-\frac{(x+1)}{2x}y^{-2}=-\frac{3}{x} \quad \text{---(1)}$$

$$\therefore z=y^{-2} \quad \text{---(2)}$$

$$\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx} \;\}\; \text{اقسم الحدود } (-2) \rightarrow \frac{1}{-2}\frac{dz}{dx}=\frac{1}{y^3}\frac{dy}{dx} \quad \text{....(3)}$$

$$-\frac{1}{2}\frac{dz}{dx}-\frac{(x+1)}{2x}z=-\frac{3}{x} \quad \}*-2$$

$$\frac{dz}{dx}+\frac{(x+1)}{x}z=\frac{6}{x} \qquad \text{by linear}$$

$$\therefore P(x)=\frac{x+1}{x} \;\&\; Q(x)=\frac{6}{x}$$

$$IF=e^{\int P(x)}\longrightarrow e^{\int\frac{x+1}{x}dx}\longrightarrow e^{\int \frac{\cancel{x}}{\cancel{x}}+\frac{1}{x}dx}$$

$$\therefore e^{\int 1+\frac{1}{x}dx}\rightarrow e^{\int x+\ln x}\xrightarrow{(10)} \underbrace{e^{x}+e^{\ln x}}_{\text{جمع}}\longrightarrow \underbrace{x\cdot e^{x}}_{\text{ضرب}}$$

جمع يتحول ← ضرب

$\therefore \; IF \cdot z = \int IF \, Q(x) \, dx + c$

$x e^{x} \cdot y^{-2} = \int \cancel{x} e^{x} \cdot \frac{6}{\cancel{x}} \, dx$

$x e^{x} \cdot y^{-2} = 6 e^{x} + c \quad \} \div x e^{x}$

$\frac{1}{y^2} = \frac{6}{x} + \frac{c}{x e^{x}}$

or $y^{-2} = \frac{6}{x} + \frac{c}{x} e^{-x}$ the general solution