CHAPTER ONE

MATRICES

1.1 Introduction

A matrix is an ordered rectangular array of numbers (or functions). For example,

$$A = \begin{bmatrix} x & 4 & 3 \\ 4 & 3 & x \\ 3 & x & 4 \end{bmatrix}$$

The numbers (or functions) are called the elements or the entries of the matrix. The horizontal lines of elements are said to constitute rows of the matrix and the vertical lines of elements are said to constitute columns of the matrix.

1.2 order of the matrix

A matrix having m rows and n columns is called a matrix of order $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). In the above example, we have A as a matrix of order 3 × 3 i.e., 3 × 3 matrix. In general, an $m \times n$ matrix has the following rectangular array:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{bmatrix}$$

The element, a_{ij} is an element lying in the i^{th} row and j^{th} column and is known as the $(i, j)^{th}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

1.3 Types of matrices

1. A matrix is said to be a row matrix if it has only one row

2. A matrix is said to be a column matrix if it has only one column.
3. A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus, an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.
4. A square matrix B = $[b_{ij}]_{n\times n}$ is said to be a diagonal matrix if it's all non-diagonal elements are zero, that is a matrix B = $[b_{ij}]_{n\times n}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.
5. A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal, that is, a square matrix B = $[b_{ij}]_{n\times n}$ is said to be a scalar matrix if
   $b_{ij} = 0$, when $i \neq j$
   $b_{ij} = k$, when $i = j$, for some constant k.
6. A square matrix in which elements in the diagonal are all 1 and rest are all zeroes is called an identity matrix.
   In other words, the square matrix $A = [a_{ij}]_{\mathrm{n}\times\mathrm{n}}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.
7. A matrix is said to be zero matrix or null matrix if all its elements are zeroes. We denote zero matrix by O.
8. Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if
   (a) they are of the same order, and
   (b) each element of A is equal to the corresponding element of B, that is, $a_{ij} = b_{ij}$ for all i and j.

1.4 Additon of Matrices Two matrices can be added if they are of the same order.

1.5 Multiplication of Matrix by a Scalar If A = $[a_{ij}]_{m\times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by a scalar k, i.e. kA = $[ka_{ij}]_{m\times n}$

1.6 Negative of a Matrix The negative of a matrix A is denoted by –A. We define –A = (–1)A.

1.7 Multiplication of Matrices The multiplication of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.

Let A = $[a_{ij}]$ be an $m \times n$ matrix and B = $[b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{th}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products i.e., $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + ... + a_{in} b_{nk}$ The matrix C = $[c_{ik}]_{m\times p}$ is the product of A and B.

Notes:

1. If AB is defined, then BA need not be defined.
2. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined if and only if $n = k$ and $l = m$.
3. If AB and BA are both defined, it is not necessary that AB = BA.
4. If the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.
5. For three matrices A, B and C of the same order, if A = B, then AC = BC, but converse is not true.
6. A. A = A2, A. A. A = A3, so on

1.8 Transpose of a Matrix

1. If A = $[a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A′ or (A^T). In other words, if A = $[a_{ij}]_{m\times n}$, then $A^T = [a_{ji}]_{n\times m}$.

2. Properties of transpose of the matrices For any matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$,

(ii) $(kA)^T = kA^T$ (where k is any constant)

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$

1.9 Symmetric Matrix and Skew Symmetric Matrix

(i) A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$, that is, $a_{ij} = a_{ji}$ for all possible values of i and j.

(ii) A square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $A^T = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j.

Note : Diagonal elements of a skew symmetric matrix are zero.

(iii) Theorem 1: For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew symmetric matrix.

(iv) Theorem 2: Any square matrix A can be expressed as the sum of a symmetric matrix and a skew symmetric matrix, that is

$$A = \frac{(A + A^T)}{2} + \frac{(A - A^T)}{2}$$

1.10 Invertible Matrices

(i) If A is a square matrix of order $m \times m$, and if there exists another square matrix B of the same order $m \times m$, such that $AB = BA = I_m$, then, A is said to be invertible matrix and B is called the inverse matrix of A and it is denoted by A–1 .

Note : 1. A rectangular matrix does not possess its inverse, since for the products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.

2. If B is the inverse of A, then A is also the inverse of B.

(ii) Theorem 3: (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

(iii) Theorem 4: If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

1.11 Inverse of a Matrix using Elementary Row or Column Operations

To find A–1 using elementary row operations, write A = IA and apply a sequence of row operations on (A = IA) till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Note: In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Example 1: if $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 8 \end{bmatrix} = 0$, find the value of x.

Solution: we have $[2x \quad 3]\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ 8 \end{bmatrix} = 0 \Rightarrow [2x - 9 \quad 4x]\begin{bmatrix} x \\ 8 \end{bmatrix} = 0$

$[2x^2 - 9x \quad 32x] = 0 \Rightarrow 2x^2 + 23x = 0 \Rightarrow x(2x+23) = 0$

$x = 0$, $x = -23/2$

Example 2: Express the matrix A as the sum of a symmetric and a skew symmetric matrix, where

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$$

Solution: we have

$$A = \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} \quad then \quad A' = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & -2 \\ -6 & 5 & 4 \end{bmatrix}$$

Hence $\frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix} 4 & 11 & -5 \\ 11 & 6 & 3 \\ -5 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 11/2 & -5/2 \\ 11/2 & 3 & 3/2 \\ -5/2 & 3/2 & 4 \end{bmatrix}$

And $\frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & -3 & -7 \\ 3 & 0 & 7 \\ 7 & -7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 & -7/2 \\ 3/2 & 0 & 7/2 \\ 7/2 & -7/2 & 0 \end{bmatrix}$

Therefore,

$$\frac{A+A'}{2} + \frac{A-A'}{2} = \begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix} = A$$

Example 3: if $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, then show that A satisfies the equation

$A^3 – 4A^2 – 3A + 11I = O$.

Solution: $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix}$

$= \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

And $\quad A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \text{ x } \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix}$

$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$

Now $A^3 – 4A^2 – 3A + 11I$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 24-36-6+0 & 34-36-9+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Example 4: let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$. Then show that $A^2 – 4A + 7I = O$. using this result calculate A^5 also.

Solution: we have $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$,

$$-4A = \begin{bmatrix} -8 & -12 \\ 4 & -8 \end{bmatrix} \text{ and } 7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Therefore, $A^2 – 4A + 7I = \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\Rightarrow$ A2 = 4A – 7I

Thus $A^3 = A.A^2 = A\,(4A - 7I) = 4\,(4A - 7I) - 7A$

$$= 16A - 28I - 7A = 9A - 28I$$

And so $A^5 = A^3\,A^2$

$$= (9A - 28I)\,(4A - 7I)$$

$$= 36A^2 - 63A - 112A + 196I$$

$$= 36\,(4A - 7I) - 175A + 196I$$

$$= -31A - 56I$$

$$= -31\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

CHAPTER FOUR

FOURIER SERIES

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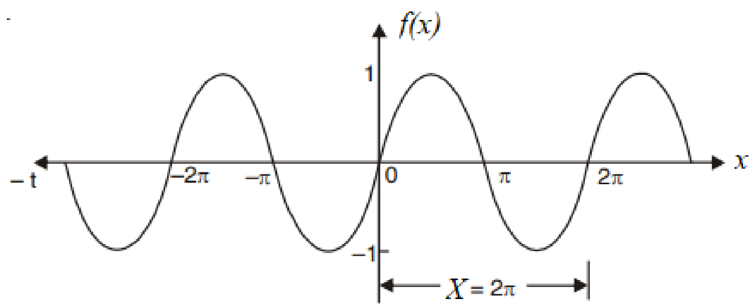
FOURIER SERIES

4.1. PERIODIC FUNCTIONS

If the value of each ordinate $f_{(x)}$ repeats itself at equal intervals in the abscissa, then $f_{(x)}$ is said to be a periodic function.

If $f_{(x)} = f_{(x + X)} = f_{(x + 2X)} = ...$ then X is called the period of the function $f_{(x)}$.

For example : sin x = sin (x + 2π) = sin (x + 4 π) = ... so sin x is a periodic function with the period 2π. This is also called sinusoidal periodic function.



4.2. FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics.

$\frac{a_o}{2}+ \; a_1 \cos x + a_2 \cos 2x + a_3 \cos 3\, x + \ldots\ldots + a_n \cos\frac{n\pi x}{L} + b_1 \sin x +$
$b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin\frac{n\pi x}{L}$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

This period depends on length factor so the period is –L ≤ x ≤ L. if the period changed to angles when L equal π, so a series of sines and cosines of an angle and its multiples of the form –π ≤ x ≤ π.

$$= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin \mathrm{nx})$$

4.3. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor’s series).

2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.

4. Fourier series of a discontinuous function is not uniformly convergent at all points.

5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

4.4. DETERMINATION OF FOURIER COEFFICIENTS

$F(x) = \frac{a_o}{2} + a_1 \cos x + a_2 \cos 2x +$

$a_3 \cos 3x + \ldots\ldots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots +$

$b_n \sin nx$ .... (1)

(i) To find a_o : Integrate both sides of (1) from x = 0 to x = 2π.

$$\int_0^{2\pi} f_{(x)}\, dx = \frac{a_o}{2}\int_0^{2\pi} dx$$

$$+ \ a_1\int_0^{2\pi} \cos x\, dx$$

$$+ a_2\int_0^{2\pi} \cos 2x\, dx$$

$$+ \ldots\ldots + a_n\int_0^{2\pi} \cos nx\, dx + b_1\int_0^{2\pi} \sin x\, dx$$

$$+ b_2\int_0^{2\pi} \sin 2x\, dx + b_n\int_0^{2\pi} \sin \mathrm{nx}\ dx$$

$$= \frac{a_o}{2}\int_0^{2\pi} dx \qquad \text{(other integrals = 0)}$$

$$\int_0^{2\pi} f(x)\, dx = \frac{a_o}{2} 2\pi \quad \text{then} \quad a_o = \frac{1}{2\pi}\int_0^{2\pi} f_{(x)}\, d_x$$

(ii) To find a_n : Multiply each side of (1) by cos nx and integrate from x = 0 to x= 2π

$$\int_0^{2\pi} f_{(x)}\, cosnx\; dx$$

$$= \frac{a_o}{2}\int_0^{2\pi} cosnx\; dx$$

$$+\; a_1\int_0^{2\pi} cosnx \cos x\; dx \ldots\ldots + a_n\int_0^{2\pi} cos^2 nx\; dx + b_1\int_0^{2\pi} \sin x cosnx\; dx$$

$$+ b_2\int_0^{2\pi} \sin 2x cosnx\; dx + b_n\int_0^{2\pi} sin^2 \text{nx}\; dx$$

$=a_n \int_0^{2\pi} cos^2 nx\; dx = a_n\pi$ (Other integrals = 0)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \cos nx\;\; d_x$$

(iii) To find b_n : Multiply each side of (1) by sin nx and integrate from x = 0 to x = 2π.

$$\int_0^{2\pi} f_{(x)} sinnx\ dx$$

$$= \frac{a_o}{2}\int_0^{2\pi} sinnx\ dx$$

$$+\ a_1\int_0^{2\pi} sinnx \cos x\ dx \ldots\ldots + a_n\int_0^{2\pi} cos\, nx\ sinnx\ dx + b_1\int_0^{2\pi} \sin x sinnx\ dx$$

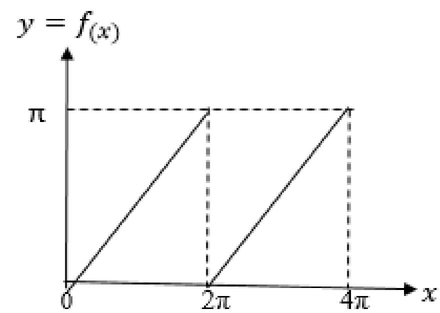
$$+ b_n\int_0^{2\pi} sin^2 \text{nx}\ dx$$

$=b_n \int_0^{2\pi} sin^2 nx\ dx = b_n\pi$ (Other integrals = 0)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \sin nx\ d_x$$

Example 1: Find the Fourier series for the periodic function shown

Solution:



First have to find the function from

straight line equation y= $m\ x$ while

$$m = \frac{y2-y1}{x2-x1}$$

$$y = \frac{\pi-0}{2\pi-o}\,x \qquad \text{then} \qquad y = \frac{x}{2} = f_{(x)}$$

Now, have to find Fourier coefficients a_o, a_n, b_n

1. $a_o = \frac{1}{2\pi}\int_0^{2\pi} f_{(x)}\,d_x = \frac{1}{2\pi}\int_0^{2\pi}\frac{x}{2}\,d_x = \frac{1}{4\pi}\,|\frac{x^2}{2}|_0^{2\pi}$

   $a_o = \frac{1}{2\pi}\,[4\pi^2 - 0]$ then $\boldsymbol{a_o = \frac{\pi}{2}}$

Integration of two functions is

$$\int u\,d_v = u.v - \int v.\,d_u$$

$u = \frac{x}{2}$ $\qquad d_v = \cos nx\,d_x$

$d_u = \frac{1}{2}d_x$ $\qquad v = \frac{sinnx}{n}$

2. $a_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)}\cos nx\;d_x = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\cos nx\;d_x$

$$a_n = \frac{1}{\pi}\left[\frac{x\sin nx}{2n}|_0^{2\pi} - \frac{1}{2n}\int_0^{2\pi}\sin nx\;d_x\right]$$

$$a_n = \frac{1}{2n\pi}\left[2\pi \underbrace{\sin 2n\pi}_{0} - 2\pi \underbrace{\sin 0}_{0}\right] - \frac{1}{2n}\left[\frac{-\underbrace{\cos n2\pi}_{-1}}{n} + \frac{\underbrace{\cos 0}_{1}}{n}\right]$$

$$\boldsymbol{a_n = 0}$$

3. $b_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \sin nx \; d_x \; = \frac{1}{\pi}\int_0^{2\pi} \frac{x}{2} \sin nx \; d_x$

$$b_n = \frac{1}{\pi}\left[\frac{-x\cos nx}{2n}\Big|_0^{2\pi} + \frac{1}{2n}\int_0^{2\pi} \cos nx \; d_x\right]$$

$$b_n = \frac{1}{2n\pi}\left[-2\pi \cos 2n\pi - 2\pi \cos 0\right] + \frac{1}{2n}\left[\frac{\sin n2\pi}{n} - \frac{\sin 0}{n}\right]$$

Integration of two functions is

$$\int u \, d_v = u.v - \int v.d_u$$

$u = \frac{x}{2}$ $\qquad d_v = \sin nx \, d_x$

$d_u = \frac{1}{2} d_x$ $\qquad v = -\frac{\cos nx}{n}$

$$\boldsymbol{b_n = \frac{-1}{n}}$$

Now, write then general equation of Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Substitute Fourier coefficients into the general equation and determine first three series

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(0 \cos nx - \frac{1}{n} \sin nx\right)$$
$$= \frac{\pi}{2} - \left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\right]$$

Example 2: Find the Fourier series for the periodic function $f_x = 3$

$0 \leq x \leq 2\pi$

Solution: have to find Fourier coefficients a_o, a_n, b_n

1. $a_o = \frac{1}{2\pi} \int_0^{2\pi} f_{(x)} \, d_x = \frac{1}{2\pi} \int_0^{2\pi} 3 \, d_x = \frac{1}{2\pi} \left| 3x \right|_0^{2\pi}$

$a_o = \frac{1}{2\pi} [6\pi - 0]$ then $\boldsymbol{a_o = 3}$

2. $a_n = \frac{1}{\pi} \int_0^{2\pi} f_{(x)} \cos nx \; d_x = \frac{1}{\pi} \int_0^{2\pi} 3 \; \cos nx \; d_x$

$= \frac{1}{\pi} \int_0^{2\pi} 3 \; \cos nx \; d_x = \frac{3}{\pi} \left| \sin \frac{nx}{n} \right|_0^{2\pi}$

$= \frac{3}{\pi} \left[\underbrace{\frac{\sin 2\pi}{n}}_{0} - \underbrace{\frac{\sin 0}{n}}_{0} \right]$

$\boldsymbol{a_n = 0}$

3. $b_n = \frac{1}{\pi} \int_0^{2\pi} f_{(x)} \sin nx \; d_x = \frac{1}{\pi} \int_0^{2\pi} 3 \; \sin nx \; d_x$

$= \frac{1}{\pi} \int_0^{2\pi} 3 \; \sin nx \; d_x = \frac{3}{\pi} \left| \frac{-\cos nx}{n} \right|_0^{2\pi} = \frac{3}{\pi} \left[-\frac{\cos 2\pi}{n} + \frac{\cos 0}{n} \right]$

$= \frac{3}{\pi} \left[-\frac{1}{n} + \frac{1}{n} \right]$ then $\boldsymbol{b_n = 0}$

Now, write then general equation of Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Substitute Fourier coefficients into the general equation and determine first three series

$f(x) = 3 + \sum_{n=1}^{\infty}(0 \cos nx + 0\ \sin nx)$ then $f(x) = 3$

<u>Example 3:</u> Find the Fourier series for the periodic function

$$f_{(x)} = \begin{cases} 1 & 0 \le x \le \pi \\ 2 & \pi \le x \le 2\pi \end{cases}$$

<u>Solution:</u> have to find Fourier coefficients a_o, a_n, b_n

1. $a_o = \frac{1}{2\pi}\int_0^{2\pi} f_{(x)}\, d_x = \frac{1}{2\pi}\left[\int_0^{\pi} 1\, d_x + \int_{\pi}^{2\pi} 2\, d_x\right]$

$$= \frac{1}{2\pi}\left[|x|_0^{2\pi} + |2x|_{\pi}^{2\pi}\right]$$

$$\boldsymbol{a_o = \frac{3}{2}}$$

2. $a_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \cos nx\ \ d_x\ = \frac{1}{\pi}\int_0^{\pi} 1\ \cos nx\ \ d_x + \int_{\pi}^{2\pi} 2 \cos nx\ \ d_x]$

$$a_n = \frac{1}{\pi}\left[\underbrace{\frac{\sin nx}{n}}_{0}|_0^{2\pi} + \left|2\underbrace{\frac{\sin nx}{n}}_{0}\right|_{\pi}^{2\pi}\right]$$

$\sin nx = 0 \quad for\ n = 1,2,3,4 \dots \dots$

Then $\boldsymbol{a_n = 0}$

3. $b_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \sin nx\ d_x\ = \frac{1}{\pi}\int_0^{\pi} 1\ \sin nx\ d_x +$
$\int_{\pi}^{2\pi} 2 \sin nx\ d_x]$

$$b_n = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\Big|_0^{2\pi} + \Big|2\frac{-\cos nx}{n}\Big|_{\pi}^{2\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-\cos n\pi}{n} + \frac{\cos 0}{n} - \frac{2\cos n2\pi}{n} + \frac{2\cos n\pi}{n}\right]$$

$$b_n = \frac{1}{\pi n}[-\cos n\pi + 1 - 2 + 2\cos n\pi]$$

$$b_n = \frac{1}{\pi n}[\cos n\pi - 1]$$

$\cos\pi = -1$ while $\cos n\pi = (-1)^n$

Therefore $b_n = \frac{(-1)^n - 1}{\pi n}$

Even number $b_n = 0$

Odd number $\boldsymbol{b_n = \frac{-2}{\pi n}}$

$(-1)^n$ has two possible solutions depends on the value of n which could be even or odd number. $\boldsymbol{b_n = \frac{-2}{\pi n}}$

Now, write then general equation of Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Substitute Fourier coefficients into the general equation and determine first three series

$$f(x) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin nx$$

$$f(x) = \frac{3}{2} - \frac{2}{\pi} \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots \dots \dots \dots .\right)$$

Example 4: Find the Fourier series for the periodic function

$$f_{(x)} = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$$

Solution: have to find Fourier coefficients a_o, a_n, b_n

1. $a_o = \frac{1}{2\pi} \int_0^{2\pi} f_{(x)}\, d_x = \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0\, d_x + \int_0^{\pi} sinx\, d_x\right]$

$$a_o = \frac{1}{2\pi} [-\cos \pi + \cos 0]$$

$$\boldsymbol{a_o = \frac{1}{\pi}}$$

2. $a_n = \frac{1}{\pi} \int_0^{2\pi} f_{(x)} \cos nx\ d_x = \frac{1}{\pi} \int_0^{\pi} \sin x\ \cos nx\ d_x$

sinx coxnx = 1/2 [sin(n+1) x – sin(n-1) x]

$$a_n = \frac{1}{2\pi} \int_0^{\pi} sinx(n+1)x - \sin(n-1)\, x] d_x$$

$$a_n = \frac{1}{2\pi} \left[\frac{-\cos(n+1)x}{n+1} + \frac{-\cos(n-1)x}{n-1} \right|_0^\pi$$

$$a_n = \frac{1}{2\pi} \left[\frac{-\cos(n+1)\pi}{n+1} + \frac{-\cos(n+1)0}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos(n-1)0}{n-1}\right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{-(-1)^{n-1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{(-1)^{n-1+1}}{n+1} + \frac{(-1)^{n-1+1}}{n-1}\right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{(-1)^{n}+1}{n+1} + \frac{(-1)^{n}+1}{n-1}\right]$$

$(-1)^n$has two possible solutions depends on the value of n which could be even or odd number.

either n is odd number so, $a_n = 0$

or n is even number so, $\boldsymbol{a_n = \frac{1}{2\pi} \left[\frac{2}{n+1} + \frac{2}{n-1}\right]}$

to avoid uncompleted solution, have to integrate for $\boldsymbol{a_1}$ and get value to substitute in Fourier series equation. So,

$$a_1 = \frac{1}{\pi} \int_0^\pi \sin x \ \cos x \ d_x = \frac{1}{2\pi} \left[sin^2 x\Big|_0^\pi\right]$$

$$\boldsymbol{a_1 = 0}$$

3. $b_n = \frac{1}{\pi} \int_0^{2\pi} f_{(x)} \sin nx \ d_x \ = \frac{1}{\pi} \int_0^{\pi} sinx \ \ sin\, nx \ d_x$

$$\boxed{\text{sinx sinnx = 1/2 [cos(n-1) x – cos(n+1) x]}}$$

$$b_n = \frac{1}{2\pi} \int_0^{\pi} [cos(n-1)\,x - cos(n+1)\,x]d_x$$

$$b_n = \frac{1}{2\pi} [\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} |_0^{\pi}]$$

$\sin nx = 0 \quad for\ n = 1,2,3,4 \ldots \ldots$

Then $\boldsymbol{b_n = 0}$

Now, write then general equation of Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Substitute Fourier coefficients into the general equation and determine first three series

$$f(x) = \frac{1}{\pi} - \sum_{n=1}^{\infty} \frac{\mathbf{1}}{\mathbf{2\pi}} [\frac{\mathbf{2}}{\boldsymbol{n+1}} + \frac{\mathbf{2}}{\boldsymbol{n-1}}] \cos nx$$

$$f(x) = \frac{1}{\pi} - \frac{1}{2\pi}\ (0 + \frac{8}{3}\ cos2x\ +\ \frac{3}{2}\ cos3x + \cdots \ldots \ldots \ldots .)$$

Example 5: Find the Fourier series for the periodic function

$$f_{(x)} = \begin{cases} \cos x & 0 \le x \le \pi \\ 0 & \pi \le x \le 2\pi \end{cases}$$

Solution: have to find Fourier coefficients a_o, a_n, b_n

1. $a_o = \frac{1}{2\pi} \int_0^{2\pi} f_{(x)}\, d_x = \frac{1}{2\pi}\ [\int_{-\pi}^{0} 0\, d_x + \int_{\pi}^{2\pi} cosx\, d_x]$

$$a_o = \frac{1}{2\pi}\ [\sin 0 + \sin -\pi]$$

$$\boldsymbol{a_o = 0}$$

2. $a_n = \frac{1}{\pi} \int_0^{2\pi} f_{(x)} \sin nx\ d_x\ = \frac{1}{2\pi} \int_{\pi}^{2\pi} cosx\ \ cos\, nx\ \ d_x$

cosx cosnx = 1/2 [cos(n+1) x + cos(n-1) x]

$$a_n = \frac{1}{2\pi} \int_{\pi}^{2\pi}\ [cos(n+1)x +\ cos(n-1)\ x]d_x$$

$$a_n = \frac{1}{2\pi}\ [\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} |_{\pi}^{2\pi}]$$

$\sin nx = 0\ \ \ for\ n = 1,2,3,4 \ldots \ldots.$

Then $\boldsymbol{a_n = 0}$

3. $b_n = \frac{1}{\pi}\int_0^{2\pi} f_{(x)} \cos nx\ d_x\ = \frac{1}{\pi}\int_{\pi}^{2\pi} cosx\ \sin nx\ d_x$

$$\boxed{\text{cosx sinnx = 1/2 [sin(n-1) x + sin(n+1) x]}}$$

$$b_n = \frac{1}{2\pi}\int_{\pi}^{2\pi} sinx(n-1)x + \sin(n+1)\,x]d_x$$

$$b_n = \frac{1}{2\pi}\,[\frac{-\cos(n-1)x}{n-1} + \frac{-\cos(n+1)x}{n+1}|_{\pi}^{2\pi}$$

$$b_n = \frac{1}{2\pi}\,[\frac{-\cos(n-1)\pi}{n-1} + \frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)0}{n-1} + \frac{\cos(n+1)0}{n+1}]$$

$$b_n = \frac{1}{2\pi}\,[\frac{-(-1)^{n-1}}{n-1} + \frac{-(-1)^{n-1}}{n+1} + \frac{1}{n-1} + \frac{1}{n+1}]$$

$$b_n = \frac{1}{2\pi}\,[\frac{(-1)^{n-1+1}}{n-1} + \frac{(-1)^{n-1+1}}{n+1}]$$

$$b_n = \frac{1}{2\pi}\,[\frac{(-1)^{n}+1}{n-1} + \frac{(-1)^{n}+1}{n+1}]$$

$(-1)^n$has two possible solutions depends on the value of n which could be even or odd number.

either n is odd number so, $b_n = 0$

or n is even number so, $\boldsymbol{b_n = \frac{1}{2\pi}\,[\frac{2}{n-1} + \frac{2}{n+1}]}$

to avoid uncompleted solution, have to integrate for $\boldsymbol{b_1}$ and get value to substitute in Fourier series equation. So,

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \cos x \;\; \cos x \;\; d_x = \frac{1}{2\pi}\int_{\pi}^{2\pi} cos2x\; d_x + \frac{1}{2}\int_{\pi}^{2\pi} dx$$

$$b_1 = \frac{1}{2\pi}\; [\underbrace{\frac{\sin 2x}{2}}_{\mathbf{0}}\Big|_{\pi}^{2\pi} + \frac{x}{2}\Big|_{\pi}^{2\pi}] = \frac{1}{2\pi}[\frac{2\pi}{2} - \frac{\pi}{2}]$$

$$\boldsymbol{b_1 = \frac{1}{4}}$$

Now, write then general equation of Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Substitute Fourier coefficients into the general equation and determine first three series

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2\pi}\; [\frac{2}{n-1} + \frac{2}{n+1}]\; \sin nx$$

$$f(x) = \frac{1}{2\pi}\; (\frac{1}{4} sinx + \frac{8}{3}\; sin2x \;+\; \frac{3}{2}\; sin3x + \cdots \ldots \ldots \ldots .)$$

Exercises: Find the Fourier series for the periodic functions

1. $f_{(x)} = \begin{cases} x & 0 \le x \le \pi \\ \pi & \pi \le x \le 2\pi \end{cases}$

2. $f(x) = x\sin x$, for $0 \le x \le 2\pi$

3. $f(x) = k$. for $0 \le x \le 2\pi$

4. $f(x) = x + x^2$, for $-\pi \le x \le \pi$

5. $f_{(x)} = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

6. $f_{(x)} = \begin{cases} 2 & 0 \le x \le 1 \\ x & 1 \le x \le 2 \end{cases}$

7. $f(x) = \sin\frac{\pi x}{l}$, for $0 \le x \le 1$