Chapter 1

Introduction To Automatic Controls

What is control: Control is the process of changing, manually or automatically, the performance of a system to be desired one. It is a series of actions directed for making a variable system adheres to a reference value (that might be either constant or variable).

Why control: Because systems by themselves usually do not behave the way we would like them to be.

Control Objectives:

1. Safety
2. Environmental Protection
3. Equipment Protection
4. Smooth Plant Operation and Production Rate
5. Product Quality
6. Profit Optimization
7. Monitoring and Diagnosis

Historical Development:

- Energy should be supplied for doing work and accomplish functions, so early people had relied upon their own brute strength or that of beasts of burden.
- Simple mechanical devices such as wheels and levers were used to accomplish immortal feats like building of high pyramids by Egyptians and Roman highways and aqueducts.
- Early people began to satisfy the increasingly growth in their demands by utilizing power from natural sources such as winds for powering sailing vessels and windmills, also waterfalls were used for turning water wheels.
- Invention of Steam engine was a milestone in human progress because it provided people with useful power that could be used at will.
- Since then, many different means have been devised for obtaining convenient sources of energy while engineers design and develop machines and equipment.
- High performance and desired output of machines and equipment could be maintained by utilizing suitable control devices while early machines used control means (controllers) of a manual nature as shown in **Figure 1**.
- Development of control engineering provides machines and equipment with high performance control devices (controllers) in which automatic controls relieve people of many monotonous activities. Also modern complex controls can perform functions beyond the physical abilities of people.
- It is well known that as applications and uses of control devices (controllers) have increased, also the demands upon the performance of these control systems increased.

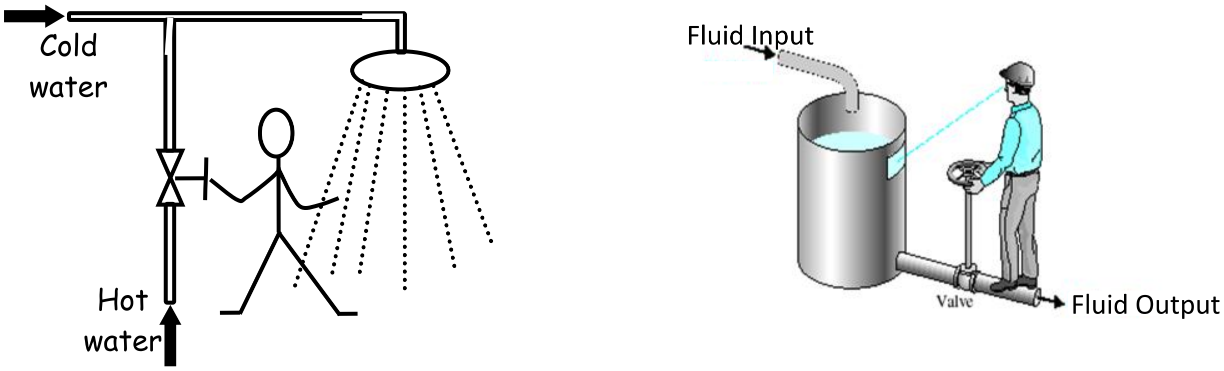
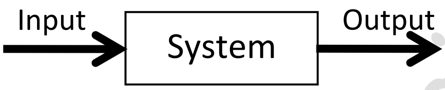


Figure 1 Manual control systems of regulating temperature of water and level of fluid in tank by adjusting input valve

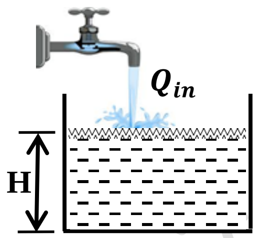
Terminology:

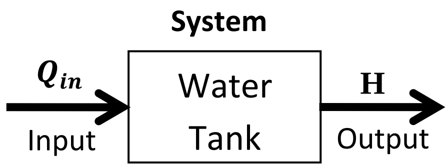
System: is a collection of components and processes coordinated together to perform a function in which system could be described as a block diagram as shown in **Figure**.



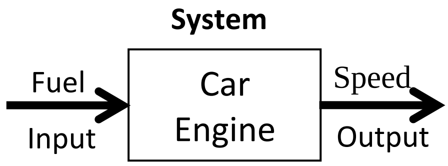
Systems may be mechanical, electrical, chemical, heating, air-conditioning, and here are some examples,

1. The process of filling tank with water is a mechanical system in which water flow rate Q_{in} is an input while $\mathbf{H}$ height of water is output as shown in **Figure**.

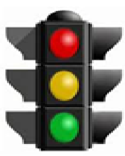


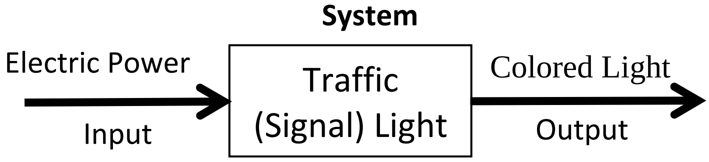


2. Process of driving a car is a mechanical system in which fuel is input while speed is output as shown in **Figure**.



3. Input of automatic traffic (signal) system at roadway intersections is the electric power and output is the colored signal light as shown in **Figure**.





Control System (System with Controller): is an interconnection or an arrangement of physical components connected in such a manner to form a system configuration that will provide a desired response. A control system is designed to achieve a specified desired purpose as shown in **Figure 2**.

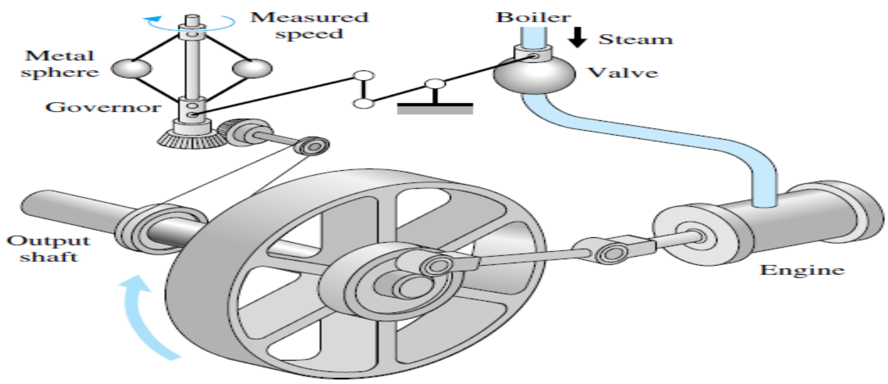
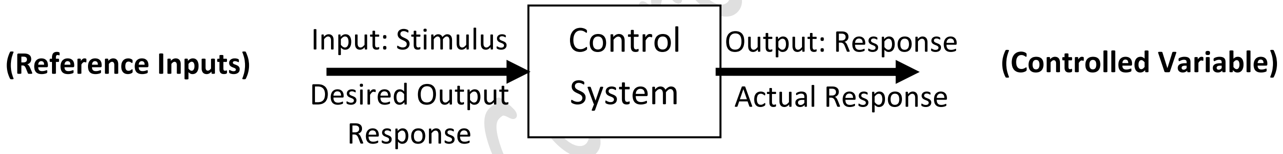


Figure 2 Watt's Fly weight governor, invented by Watt in 18th Century used to control speed of the steam engine

In order to identify or define a control system, two terms should be introduced:

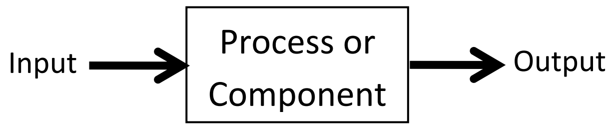
1. Input: is the stimulus, excitation or command signal applied to control system. Inputs could be physical variables or abstract ones such as reference, set point or desired values for the output of the control systems.
2. Output: is the actual response resulting from a control system in which it may or may not be equal to the specified response implied by the input.

Input and output represent desire and the actual response respectively as shown in **Figure**.

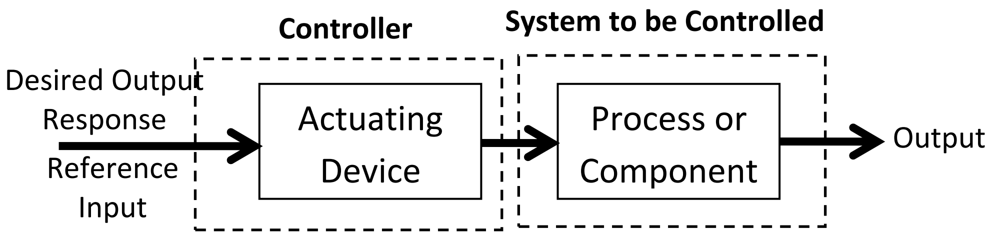


Control System Configurations:

1. **Open-loop control system:** A control type that computes its input into a system using only the current state. It does not use feedback to determine whether its output has achieved the expected goal. An open-loop control system utilizes a controller or control actuator to obtain the desired response as shown in **Figure**.



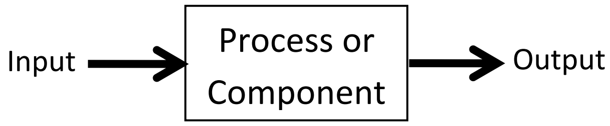
(Process or Component to be Controlled**)**



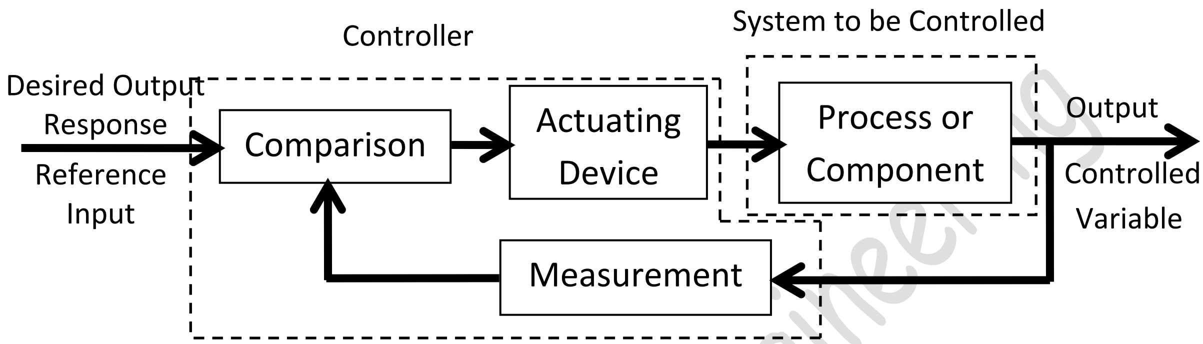
(Open-loop Control System without Feedback**)**

There are many examples of open-loop control system such as electric ovens, gas ring, water tap, traffic signals, washing machine, electric fan, irrigation sprinkler system, electric motor, and manual operation of the accelerator in automobile.

Closed-loop control system: A type of control system that manually or automatically changes the output according to the difference between feedback and input signals. In contrast to an open-loop control system, a closed-loop utilizes an additional measure of the actual output to compare with the desired output response as shown in **Figure**.



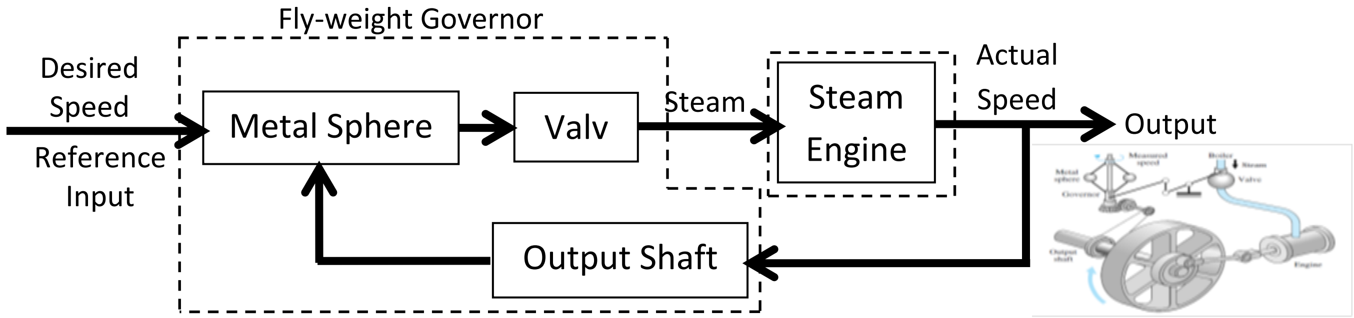
(Process or Component to be Controlled**)**



Closed-Loop Feedback Control System (with feedback)

Examples of closed loop control system such as heating and air-conditioning systems, DC servo motor, speed control system, filling tank with water using float, tension-regulating apparatus used in paper industry.

Feedback Control System: A control system in which the value of some output quantity is controlled by measuring and feeding back the value of it so as to bring this value closer to a desired one. Also it is known as closed-loop control system such as speed control system which is shown in Figure.



Closed-Loop Feedback Speed Control System (with feedback)

<u>**Effect of feedback control system:**</u>

- Reduce Error between the actual and desired value.
- Change the stability of the system.
- Change overall system gain.
- Change sensitivity of the system gain.
- Reduce effects of external disturbance.
- Reduce effect of variations of system parameters.

<u>**Classification of Feedback Control System:**</u>

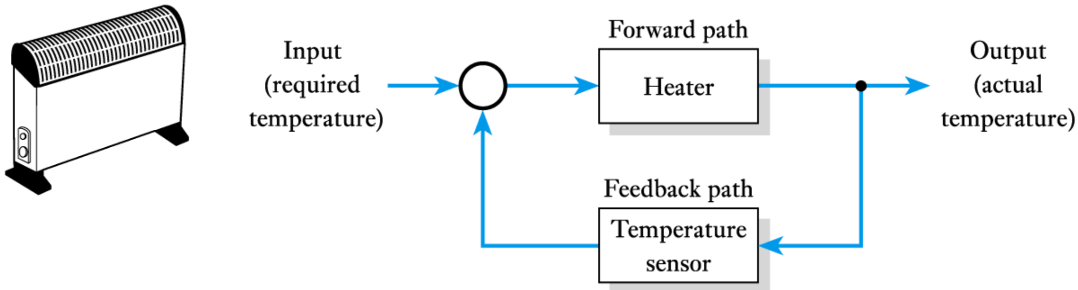
Feed back control systems are divided into two classes:

1. **Servomechanism:** A *servomechanism* is a power-amplifying feedback control system in which the controlled variable is a mechanical position or a time derivative of position such as velocity or acceleration. An automatic aircraft landing system is an example of servomechanism.
2. **Regulators.** A *regulator* is a feedback control system in which the reference input or command is constant for long periods of time such input is known as *set point*. Most temperature controllers are regulators.

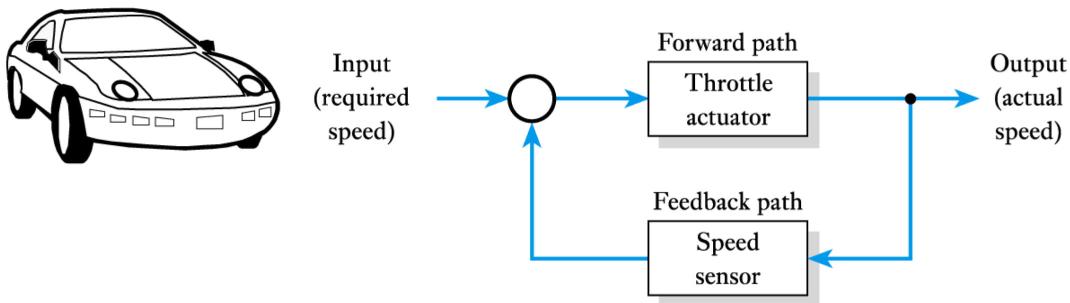
<u>Examples of Control Systems:</u>

Many applications could be found for control systems in science, industry, and home. Here are a few examples:

1. xResidential heating and air-conditioning systems controlled by a thermostat (closed-loop control system) as shown in **Figure.**

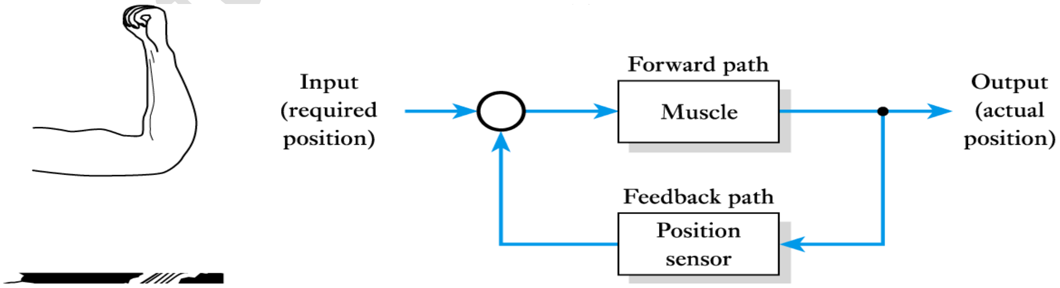


2. The cruise (speed) control of an automobile (closed-loop control system) as shown in **Figure.**



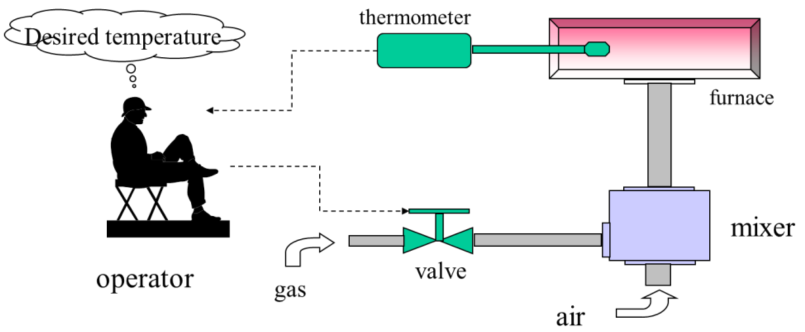
3. Position control in a human lib (closed-loop control system) as shown in **Figure.**

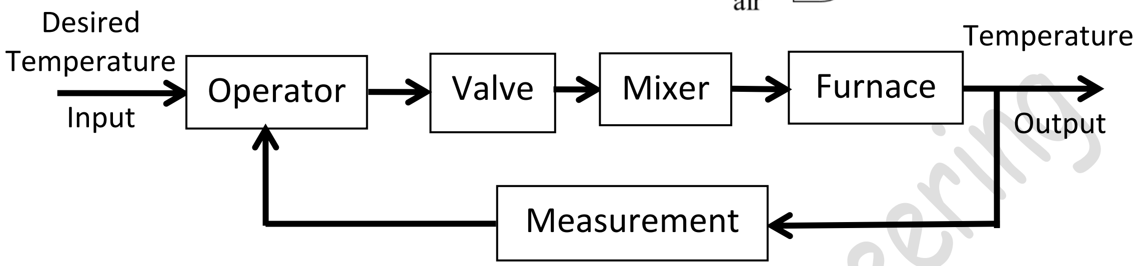
4.



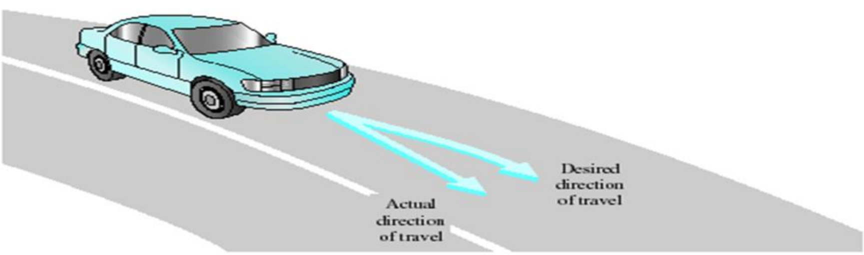
5.
6.
7.
8.

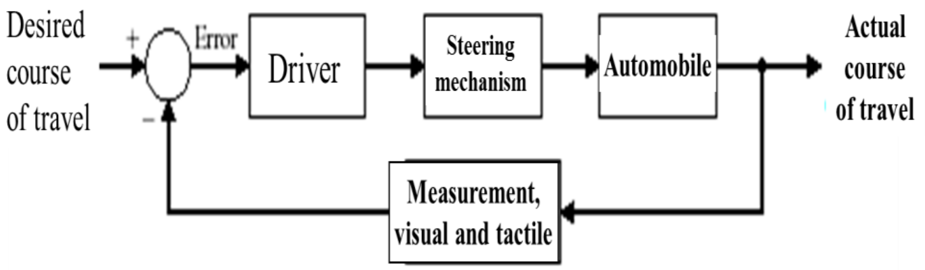
4. Temperature control system (closed-loop control system) as shown in **Figure.**



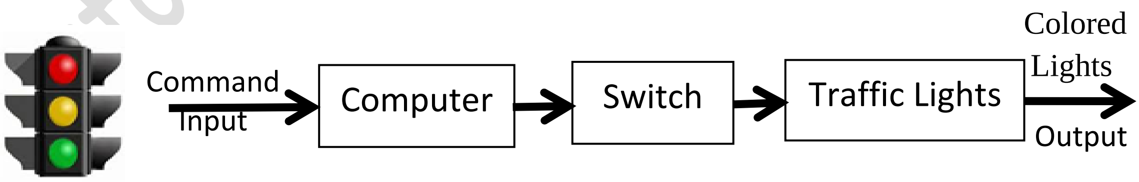


5. Automobile steering control system (closed-loop control system) as shown in **Figure**.





6. Control of traffic lights (open-loop control system) as shown in **Figure**.



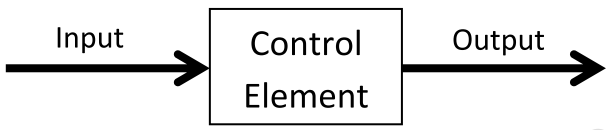
7. Activation of a light switch to regulate the illumination in a room
8. Human controlling the speed of an automobile by regulating gas supply to the engine
9. Automatic traffic control (signal) system at roadway intersections
10. Automatically control system turns on a room lamp at dusk, and turns it off in day light
11. Automatic hot water heater
12. Environmental test-chamber temperature control system
13. An automatic positioning system for a missile launcher
14. An automatic speed control for a field-controlled dc motor
15. The attitude control system of a typical space vehicle

16. Automatic position-control system of a high speed automated train system
17. An elevator-position control system used in high-rise multilevel buildings.

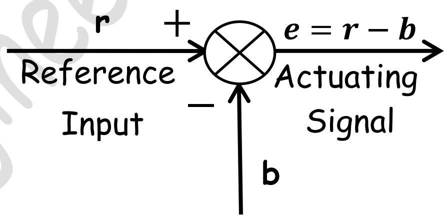
<u>System Representation</u>:

Mathematical relationships of control elements are usually represented by block diagram where inputs and outputs are described by arrows. Overall block diagram representation for an entire system can be constructed by combining block diagrams for each component of the system.

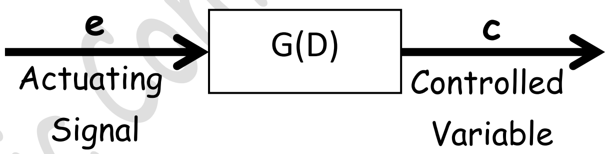
- **Block:** Any component or process to be controlled can be represented by a block which is described as a box refers to multiplication operation. A block may be a set of elements described by an input/output relationship as shown in **Figure**.



- **Comparator**: is represented by a circle refers to the summing operation which makes a comparison between feedback signal (b) and reference input (r) while an actuating signal (e) may be existed according to the difference between feedback signal (b) and reference input (r) as shown in **Figure**.



- **Control Elements:** Portion of a control system between the actuating signal (e) and the controlled variable (c) is called the control elements as shown in **Figure**.

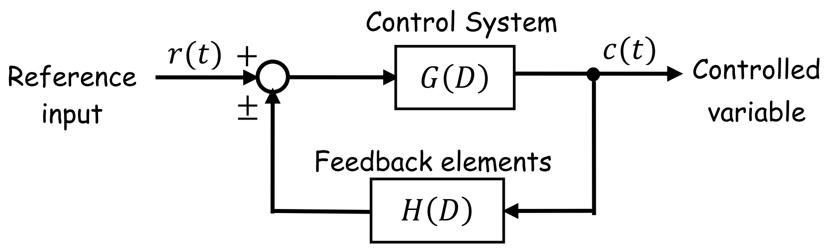


And, $c = G(D)e$

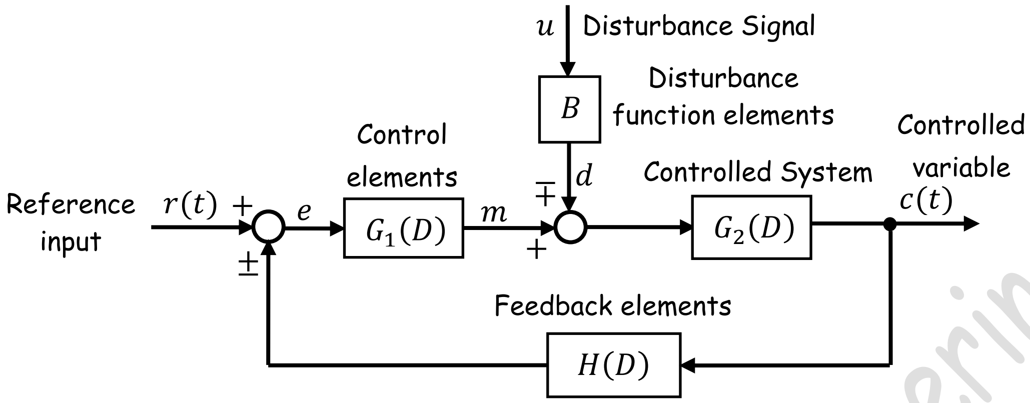
Where, G(D) represents mathematical differential equations of operation that represents input and output relationship. Also it is called as the transfer function and could be calculated as,

$$G(D) = \frac{Output}{Input}$$

- **Complete block diagram**: complete over all block diagram representation for the whole control system with single input single output (SISO) could be described generally as shown in **Figure**.



Where, H(D) represents mathematical differential equations of operation for feedback elements. Also complete over all block diagram representation for the whole control system with two inputs single output (MISO) could be described generally as shown in **Figure**.



Chapter 2

REPRESENTATION OF CONTROL COMPONENTS

To investigate the performance of control systems, it is necessary to obtain the mathematical relationship $G(D)$ relating the controlled variable $c(t)$ and the actuating signal $e(t)$ of the forward elements. This is accomplished by first obtaining the mathematical representation for each component and then expressing each of these equations as a block diagram.

Operational Notation:

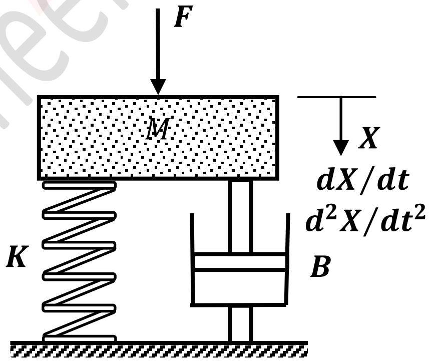
Operator D is a symbol which indicates differentiation with respect to time,

- $D^n = \frac{d^n}{dt^n}$ $n = 1,2,3, \ldots..$
- $D(x+y) = \frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = Dx + Dy$
- $\int dt = \frac{1}{D} = Integration$

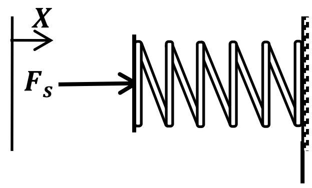
Representation of Mechanical Components:

1. **Mass-Spring Damper:**

Mass-spring damper shown in **Figure** is a mechanical control component used to control oscillations and vibrations in machines and equipment. In which the applied total force $\boldsymbol{F}$ is input while the displacement $\boldsymbol{X}$, $\boldsymbol{dX/dt}$, $\boldsymbol{d^2X/dt^2}$, is output. Its block diagram should be represented by investigating the individual components as,



❖ **Spring**: The spring force $\boldsymbol{F_s}$ required to resist deflection a distance $\boldsymbol{X}$ is,

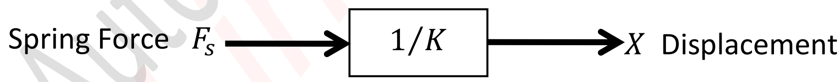


$$F_s = KX$$

Where, $\boldsymbol{K}$ is stiffness or spring rate. Since $\boldsymbol{F_s}$ is input, $\boldsymbol{X}$ is output, and $\boldsymbol{K}$ is the mechanical impedance,

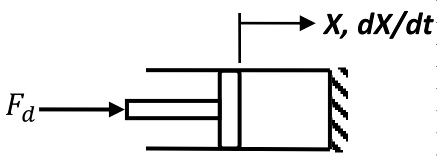
$$X = \frac{1}{K} F_s$$

The block diagram representation for the spring is,



❖ **Viscous damper:** Damping force F_d required to move one end of dashpot at velocity $\boldsymbol{V}$ is,

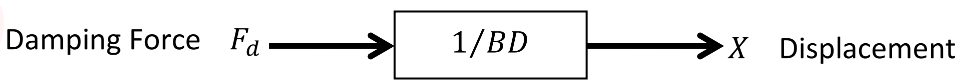
$$F_d = BV = B\frac{dX}{dt} = BDX$$



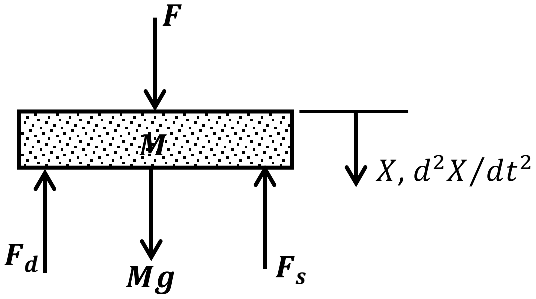
Where, B is damping coefficient of viscous damper. Since $\boldsymbol{F_d}$ is input, $\boldsymbol{X}$ is output, and $\boldsymbol{BD}$ is the mechanical impedance,

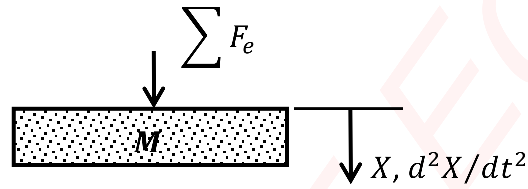
$$X = \frac{1}{BD} F_d$$

The block diagram representation for the viscous damper is,



❖ **Mass:** Using Newton's second law of motion, summation of external forces acting on a mass,



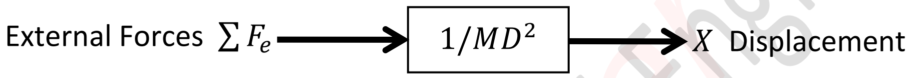


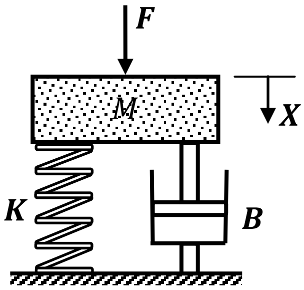
$$\sum F_e = F + Mg - F_d - F_s = M\frac{d^2X}{dt^2} = MD^2X$$

Since, $\sum F_e$ is input, deflection X is output, and MD^2 is mechanical impedance,

$$X = \frac{1}{MD^2}\sum F_e$$

The block diagram representation for mass is,





❖ For the whole mass-spring damper combination shown in **Figure**, input is total applied F while X is output,

$$\sum F_e = F + Mg - F_d - F_s = MD^2X$$
$$F = (MD^2 + BD + K)X - Mg$$

Some ways are used to represent block diagram of the whole mass-spring damper combination,

1. Force F_i required to maintain the mass at the reference position X_i

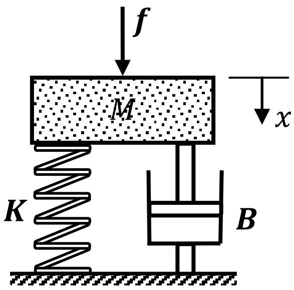
   $F_i = (MD^2 + BD + K)X_i - Mg,$

   $F - F_i = (MD^2 + BD + K)(X - X_i) - Mg + Mg$

   Variation in total force from the reference position is:

   $F - F_i = f$ also, $X - X_i = x$, then,

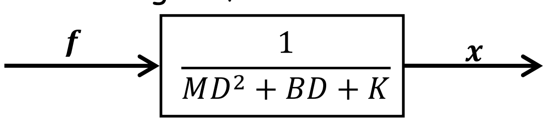
   $$f = (MD^2 + BD + K)x = Zx$$

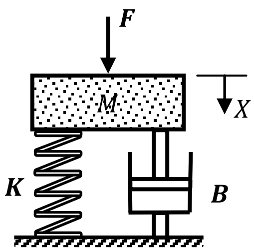
   Where, $Z = MD^2 + BD + K$, refers to the mechanical impedance of the mass-spring damper system. As shown in **Figure**, f is input and x is output,

   $$x = \frac{1}{Z}f \qquad x = \frac{1}{MD^2 + BD + K}f$$

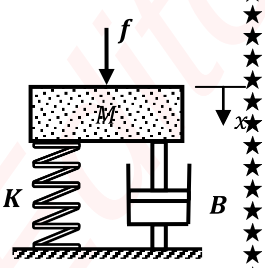
   And the associated Block diagram,

2. Since, $F = (MD^2 + BD + K)X - Mg$, using techniques of linearization about reference position,



Since, $F = F(X)$ linearization at reference position yields,



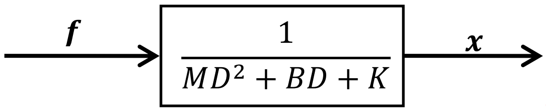
$$\Delta F = \frac{\partial F}{\partial X}\bigg|_i \Delta X, \qquad F - F_i = \frac{\partial F}{\partial X}\bigg|_i (X - X_i), \qquad f = \frac{\partial F}{\partial X}\bigg|_i x$$

$$\frac{\partial F}{\partial X}\bigg|_i = \frac{d}{dX}[(MD^2 + BD + K)X - Mg]\bigg|_i = MD^2 + BD + K$$

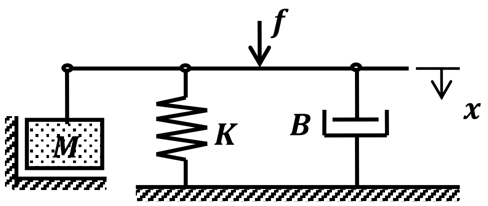
Then,

$f = (MD^2 + BD + K)x$ **Or,** $x = \frac{1}{MD^2+BD+K} f$

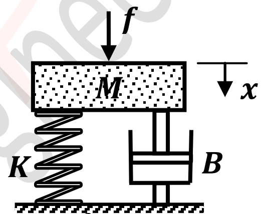
And the associated Block diagram,



3. Combination of Mass-spring damper could be easily recognized using the technique of Grounded-chair Representation as shown in **Figure**,



Equivalent Grounded-chair Representation of Mass Spring Damper Combination

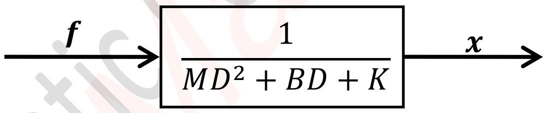


System of Mass-Spring Damper Combination

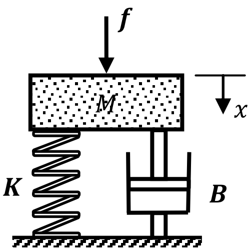
For series combination,

$f = f_1 + f_2 + f_3 \qquad f = MD^2x + BDx + Kx \qquad x = \frac{1}{MD^2+BD+K} f$

And the associated Block diagram,

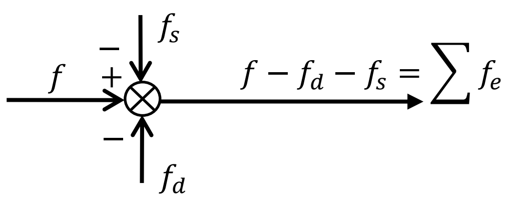


4. The whole Mass-spring damper block diagram could be represented using the techniques of block-diagram Reduction (Algebra). For the whole mass-spring damper combination shown in **Figure**,

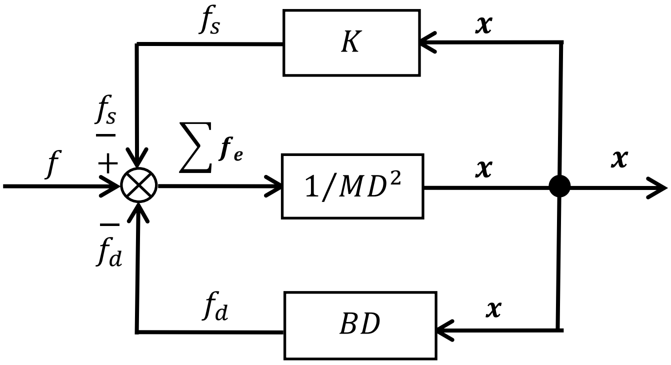


$$\sum f_e = f - f_d - f_s = MD^2x$$

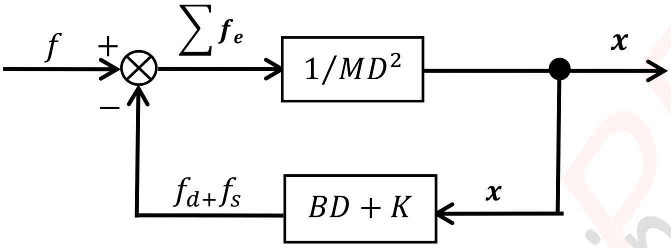
In which it could be represented as a block diagram,



The overall block diagram representation for the Torsional inertia-spring damper combination is constructed as shown in **Figure.**



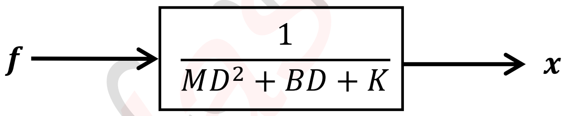
Using the rule of combining blocks in parallel,



For single input single output control system SISO, the mathematical differential equation of operation could be obtained as,

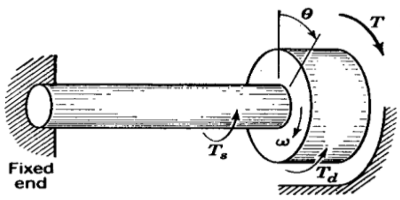
$$\frac{x}{f} = \frac{1/MD^2}{1 + \frac{BD+K}{MD^2}} = \frac{1}{MD^2 + BD + K}$$

Then,

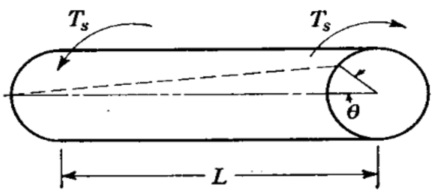


2. **Rotational Mechanical Components:**

A disk rotating in a viscous medium and supported by a shaft shown in **Figure,** is considered as Torsional inertia-spring damper combination in which the torque $\boldsymbol{T}$ is input and the angular displacement $\boldsymbol{\theta}$ is the output.

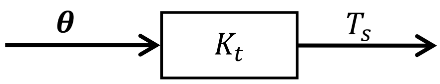


❖ **Rotating Shaft:** A rotating shaft usually behaves as a torsional spring which is subjected to a twisting torque T_s when it is displaced by an angle θ as shown in **Figure,**



$$T_s \propto \theta \qquad\qquad T_s = K_t\theta$$

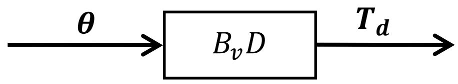
Where, K_t is a torsional spring rate. Since θ is input and T_s is output, the block diagram representation for this rotating shaft (torsional spring) is,



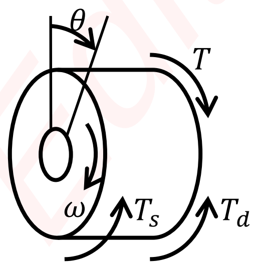
❖ **Viscous Medium**: Damping Torque $\boldsymbol{T_d}$ required to overcome effect of viscous friction in viscous medium with B_v coefficient on a rotating member,

$$T_d = B_v\omega = B_v\frac{d\theta}{dt} = B_vD\theta$$

Since θ is input and T_d is output, the block diagram representation for the process of viscous friction is,



- **Rotating Disk:** The summation of external torques $\sum \boldsymbol{T_e}$ required to rotate a disk with angular velocity ω as shown in **Figure**,

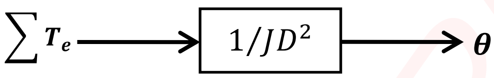


$$\sum \boldsymbol{T_e} = J\alpha = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2} = JD^2\theta$$

J is the mass moment of inertia. Since, $\sum \boldsymbol{T_e}$ is input and θ is output

$$\theta = \frac{1}{JD^2}\sum \boldsymbol{T_e}$$

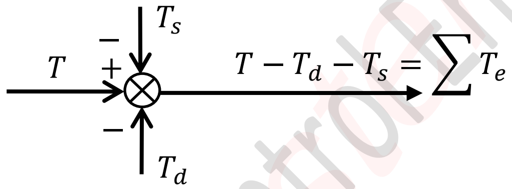
Block diagram representation of rotating disk is shown in **Figure,**



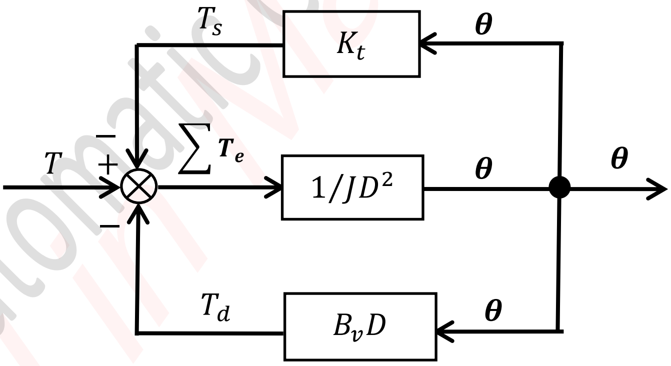
- Summation of external torques acting on the disk,

$$\sum T_e = T - T_d - T_s$$

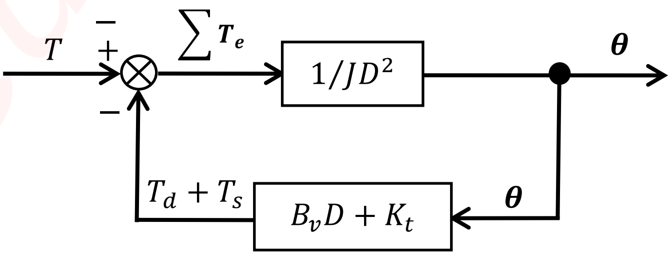
In which it could be represented as a block diagram as shown,



The overall block diagram representation for the Torsional inertia-spring damper combination is constructed as shown in **Figure**.



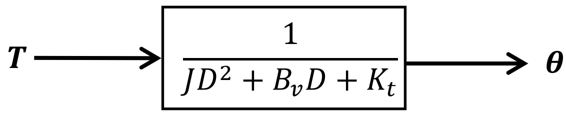
Using the rule of combining blocks in parallel,



For single input single output control system SISO, the mathematical differential equation of operation could be obtained as,

$$\frac{\theta}{T}=\frac{1/JD^2}{1+\frac{B_vD+K_t}{JD^2}}=\frac{1}{JD^2+B_vD+K_t}$$

Since, T is input and θ is output,



And the transfer function is,

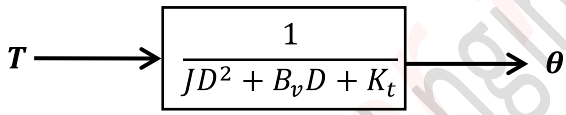
$$\frac{\theta}{T}=\frac{1}{JD^2+B_vD+K_t}$$

The overall block diagram representation for the Torsional inertia-spring damper combination may be also constructed as,

$$\sum T_e=T-T_d-T_s=J\propto=JD^2\theta \qquad T=(JD^2+B_vD+K_t)\theta=Z\theta$$

$$\theta=\frac{1}{Z}T \qquad \theta=\frac{1}{JD^2+B_vD+K_t}T$$

The overall block diagram representation as shown in **Figure**.



And the transfer function is,

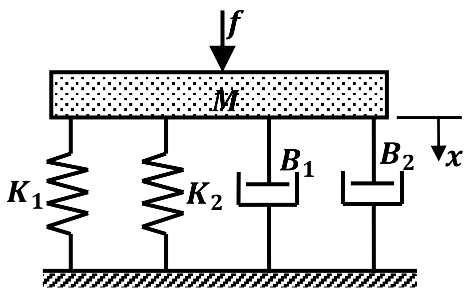
$$\frac{\theta}{T}=\frac{1}{JD^2+B_vD+K_t}$$

Series Mechanical Elements:

1. Total force is equal to the summation of forces acting on each individual component,
$$f=f_1+f_2+f_3+f_4+\cdots$$
2. Each element undergoes the same displacement,
$$x=x_1=x_2=x_3=x_4=\cdots$$
3. Equivalent mechanical impedance is equal to the summation of impedances for each individual component,
$$Z=Z_1+Z_2+Z_3+Z_4+\cdots$$

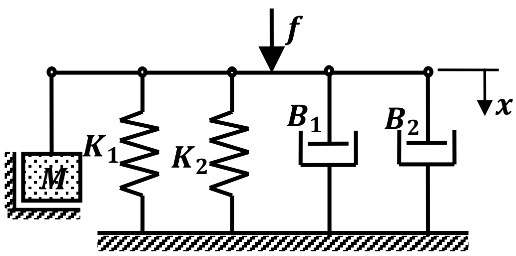
Example1: For the system of mass-spring damper combination shown in **Figure**, determine,

1. The equation relating force f and displacement x
2. The total impedance Z of mechanical system
3. Represent the associated overall block diagram



Solution:

Equivalent Grounded-chair Representation is presented for the system to be easily recognized as shown in **Figure**.



For series arrangement,

$$f=f_1+f_2+f_3+f_4+f_5$$

$$f=(MD^2+K_1+K_2+B_1D+B_2D)x=Zx$$

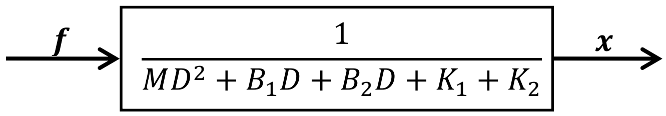
Where, Z is the equivalent impedance,

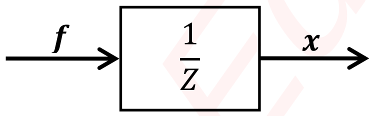
$$Z=MD^2+B_1D+B_2D+K_1+K_2$$

Since f is input and x is output,

$$x = \frac{1}{MD^2 + B_1D + B_2D + K_1 + K_2} f \qquad\qquad x = \frac{1}{Z} f$$

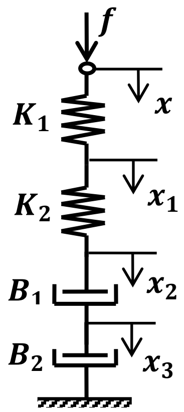
Overall block diagram representation for this system,





Parallel Mechanical Elements:

1. Same force is transmitted through each element.
$$f = f_1 = f_2 = f_3 = f_4 = \cdots$$
2. Total deflection is the sum of individual deflections of each element.
$$x = x_1 + x_2 + x_3 + x_4 + \cdots$$
3. Total impedance Z is equal to one divided by the sum of reciprocal of the individual impedances for each element.
$$Z = \frac{1}{1/z_1 + 1/z_2 + 1/z_3 + \cdots}$$



Example 2: For the system shown in **Figure**, determine,

1. The total impedance $\boldsymbol{Z}$ of mechanical system
2. The equation relating force $\boldsymbol{f}$ and displacement $\boldsymbol{x}$
3. Represent the associated overall block diagram

Solution,

Since the system is a parallel combination of mechanical elements,

$$Z_1 = K_1, \qquad Z_2 = K_2, \qquad Z_3 = B_1D, \qquad Z_4 = B_2D$$

Total impedance,

$$Z = \frac{1}{1/z_1 + 1/z_2 + 1/z_3 + 1/z_4} = \frac{1}{1/K_1 + 1/K_2 + 1/B_1D + 1/B_2D}$$

Second way,

$$x = (x - x_1) + (x_1 - x_2) + (x_2 - x_3) + (x_3 - 0)$$

$$x = \frac{f}{K_1} + \frac{f}{K_2} + \frac{f}{B_1D} + \frac{f}{B_2D} = \left(\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{B_1D} + \frac{1}{B_2D}\right) x$$

$$f = \frac{1}{1/K_1 + 1/K_2 + 1/B_1D + 1/B_2D} x = Zx$$

Total Impedance,

$$Z = \frac{1}{1/K_1 + 1/K_2 + 1/B_1D + 1/B_2D}$$

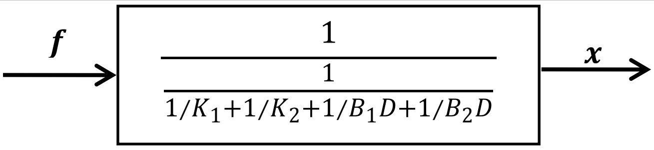
Since, $\quad f = Zx$

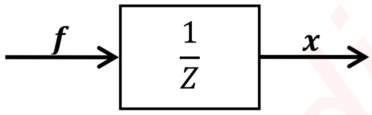
$$f = \frac{1}{1/K_1 + 1/K_2 + 1/B_1D + 1/B_2D} x$$

f is input and x is output,

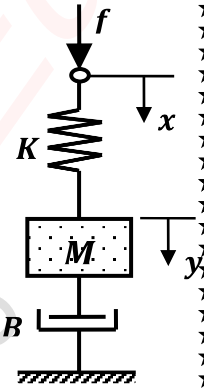
$$x = \frac{1}{Z} f \qquad\qquad x = \frac{1}{\frac{1}{1/K_1 + 1/K_2 + 1/B_1D + 1/B_2D}} f$$

Overall block diagram representation for this system,





Note: In parallel elements, same force is transmitted through each one. Springs and dampers satisfy this condition because force is same on both sides. But it is not for mass because difference in forces acting on both sides as shown in **Figure a**. Thus a mass located between other elements cannot be in parallel with them. It can be in parallel only when it is located as a last element as shown in **Figure b**.



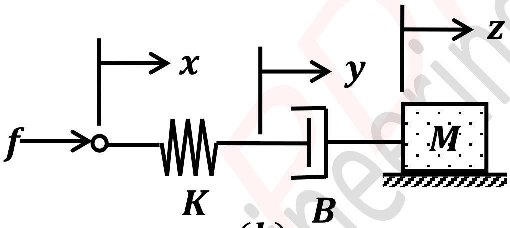
(a)

For the system in **Figure b**, the displacement x is:

$x = x_1 + x_2 + x_3$

$x = (x - y) + (y - z) + z$

$$x = \left(\frac{1}{K} + \frac{1}{BD} + \frac{1}{MD^2}\right) f$$

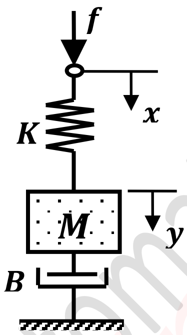


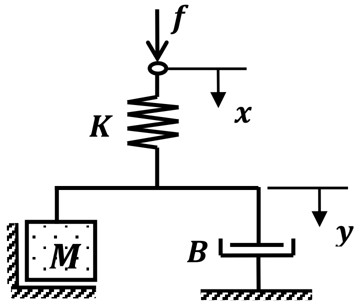
(b)

$$f = \frac{1}{1/K + 1/BD + 1/MD^2} x \qquad f = Zx \qquad Z = \frac{1}{1/K + 1/BD + 1/MD^2}$$

For system shown in **Figure a**, from the equivalent Grounded-chair Representation spring K is in parallel with the series combination of M and B.

$$Z_1 = K, \qquad Z_2 = MD^2 + BD$$





Grounded-Chair Representation

Total impedance,

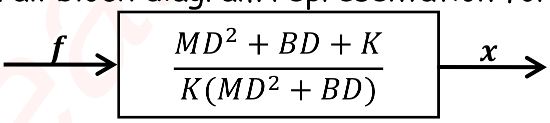
$$Z = \frac{1}{1/Z_1 + 1/Z_2} = \frac{1}{1/K + 1/(MD^2 + BD)}$$

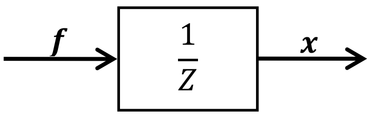
$$f = Zx = \frac{1}{1/K + 1/(MD^2 + BD)} x \qquad f = \frac{K(MD^2 + BD)}{MD^2 + BD + K} x$$

Since f is input and x is output,

$$x = \frac{MD^2 + BD + K}{K(MD^2 + BD)} f \qquad x = \frac{1}{Z} f$$

Overall block diagram representation for this system,





Note: to determine the relationship between displacements x and y, for the series portion of the Ground- Chair Representation,

$$f = f_1 + f_2 = MD^2y + BDy = (MD^2 + BD)y$$

$$\frac{K(MD^2 + BD)}{MD^2 + BD + K}x = (MD^2 + BD)y \qquad y = \frac{K}{MD^2 + BD + K}x$$

Or,

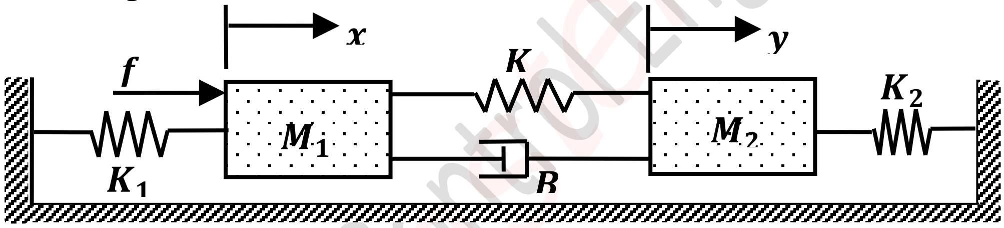
$$x = \frac{MD^2 + BD + K}{K}y \qquad x = \left(\frac{1}{K}\right)(MD^2 + BD + K)y$$

Grounded-Chair Representation:

General procedure to construct the Grounded-chair Representation is,

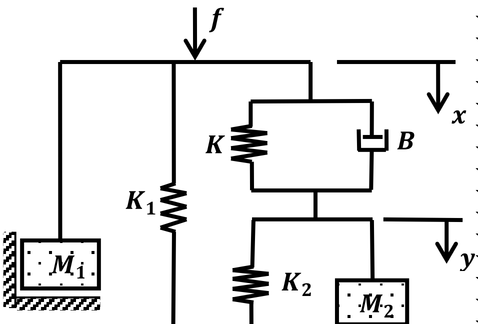
1. Identify the coordinates of the system.
2. Draw coordinates such that coordinate at which force acts is at the top and ground is at the bottom.
3. Insert each individual element in its correct location with respect to these coordinates.

Example 3: For mechanical system shown in **Figure** construct the equivalent Grounded-Chair Representation and determine equations relate f and x, x and y. Then represent the associated block diagram.



Solution:

Equivalent Grounded-Chair representation as shown in **Figure**,



$$Z = Z_1 + Z_2 + Z_3 \qquad Z_1 = M_1D^2 \qquad Z_2 = K_1,$$

$$Z_3 = \frac{1}{1/(K+BD) + 1/(K_2 + M_2D^2)} = \frac{(K+BD)(K_2 + M_2D^2)}{K_2 + M_2D^2 + K + BD}$$

Also, z_3 can be determined in such a way:

$$x = (x - y) + y$$

$$x = \frac{f_3}{K + BD} + \frac{f_3}{K_2 + M_2D^2} = \left(\frac{1}{K + BD} + \frac{1}{K_2 + M_2D^2}\right)f_3$$

$$f_3 = \frac{1}{1/(K + BD) + 1/(K_2 + M_2D^2)}x \qquad f_3 = z_3x$$

$$z_3 = \frac{1}{1/(K + BD) + 1/(K_2 + M_2D^2)} = \frac{(K + BD)(K_2 + M_2D^2)}{K_2 + M_2D^2 + K + BD}$$

$$Z = \left[M_1D^2 + K_1 + \frac{1}{1/(K + BD) + 1/(K_2 + M_2D^2)}\right] \qquad f = Zx$$

$$f = \left[M_1D^2 + K_1 + \frac{1}{1/(K + BD) + 1/(K_2 + M_2D^2)}\right]x$$

From the equivalent Grounded-Chair Representation,

$$f_3 = (K_2 + M_2D^2)y \qquad also, \qquad f_3 = \frac{1}{1/(K+BD) + 1/(K_2 + M_2D^2)}x$$

$$\frac{x}{1/(K+BD)+1/(K_2+M_2D^2)} = (K_2 + M_2D^2)y$$

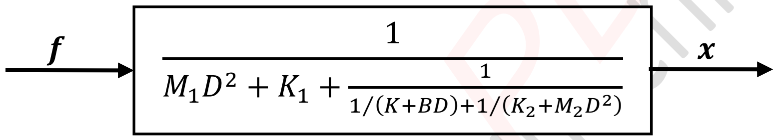
Then, equation relates x and y,

$$x = \left(1 + \frac{K_2 + M_2D^2}{K + BD}\right)y$$

From f equation above, input is f and x output then,

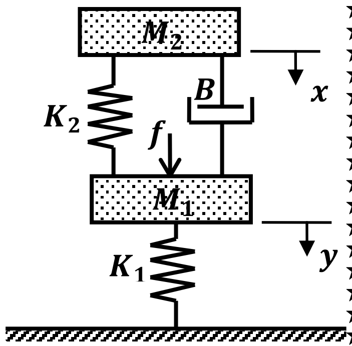
$$x = \left[\frac{1}{M_1D^2 + K_1 + \frac{1}{1/(K+BD)+1/(K_2+M_2D^2)}}\right]f$$

Overall block diagram representation,



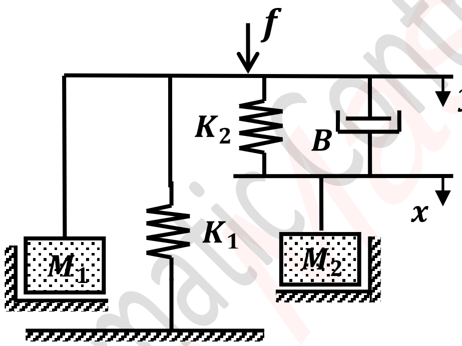
$$\frac{1}{M_1D^2 + K_1 + \frac{1}{1/(K+BD)+1/(K_2+M_2D^2)}}$$

Example 4: For mechanical system shown in **Figure**, construct the equivalent Grounded-Chair Representation and determine the equations relate f and y, x and y.



Solution:

Equivalent Grounded-Chair representation as shown in **Figure**,



$$Z_1 = M_1D^2, \qquad Z_2 = K_1, \qquad Z_3 = \frac{1}{1/(K_2 + BD) + 1/M_2D^2} = \frac{M_2D^2(K_2 + BD)}{M_2D^2 + BD + K_2}$$

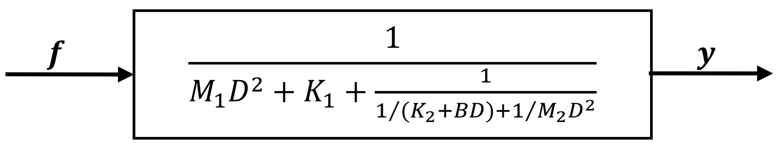
Since, $Z = Z_1 + Z_2 + Z_3$, then,

$$Z = M_1D^2 + K_1 + \frac{1}{1/(K_2 + BD) + 1/M_2D^2}$$

Since $f = Zy$, then equation relates f and y,

$$f = \left[M_1D^2 + K_1 + \frac{1}{1/(K_2 + BD) + 1/M_2D^2}\right]y$$

The associated block diagram representation is,



$$\frac{1}{M_1D^2 + K_1 + \frac{1}{1/(K_2+BD)+1/M_2D^2}}$$

For parallel portion,

$y = (y - x) + x$

$y = \frac{f_3}{K_2+BD} + \frac{f_3}{M_2D^2}$ and $y = \left(\frac{1}{K_2+BD} + \frac{1}{M_2D^2}\right) f_3$ Parallel Combination

$$f_3 = \frac{1}{1/(K_2+BD) + 1/(M_2D^2)} y = \frac{M_2D^2(K_2+BD)}{M_2D^2+BD+K_2} y$$

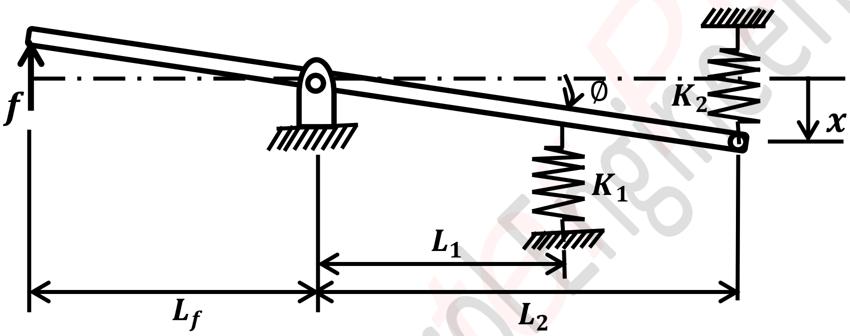
Also, $f_3 = M_2D^2x$

Then, $\frac{M_2D^2(K_2+BD)}{M_2D^2+BD+K_2} y = M_2D^2x$ $\qquad y = \left(1 + \frac{M_2D^2}{K_2+BD}\right)x$

Example 5: For the lever shown in **Figure**, the variation in the applied force is f and the variation in spring position is x. The horizontal line represents the reference position of the lever.

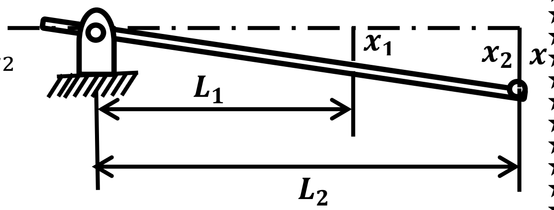
a) Determine the equation relating f and x

b) Determine the relationship between t and $\emptyset$ (where $t = fL_f$ is the variation in applied torque and, $x = L_2\emptyset$).



Solution:

a) Take the moments about the pivot position,



$\sum M_o = 0$, $\quad fL_f - f_1L_1 - f_2L_2 = 0$, $\quad fL_f = f_1L_1 + f_2L_2$

Since, $f_1 = K_1x_1$ and, $f_2 = K_2x_2$

$$fL_f = K_1x_1L_1 + K_2x_2L_2$$

From the Figure, $x_1 = \frac{L_1}{L_2}x_2$

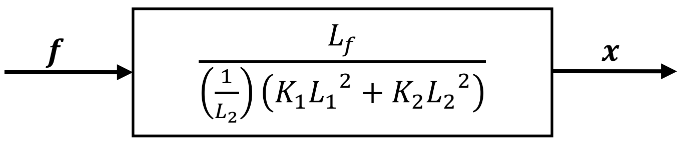
Since, $x_2 = x$, then, $x_1 = \frac{L_1}{L_2}x$, $\qquad fL_f = K_1\frac{L_1}{L_2}xL_1 + K_2xL_2 = \left(K_1\frac{L_1}{L_2}L_1 + K_2L_2\right)x$

$$f = \frac{\left(\frac{1}{L_2}\right)\left(K_1{L_1}^2 + K_2{L_2}^2\right)}{L_f}x \qquad Z = \frac{\left(\frac{1}{L_2}\right)\left(K_1{L_1}^2 + K_2{L_2}^2\right)}{L_f}$$

Since f is input and x is output,

$$x = \frac{L_f}{\left(\frac{1}{L_2}\right)\left(K_1{L_1}^2 + K_2{L_2}^2\right)} f$$

And the associated block diagram is,



$$\frac{L_f}{\left(\frac{1}{L_2}\right)\left(K_1{L_1}^2 + K_2{L_2}^2\right)}$$

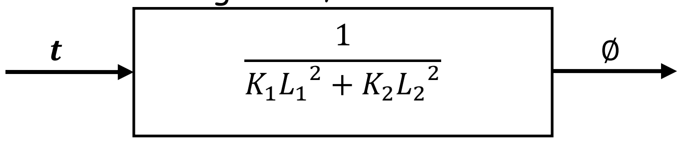
b) Since, $t = fL_f$ and $x = L_2\emptyset$

$fL_f = \left(K_1{L_1}^2 + K_2{L_2}^2\right)\frac{x}{L_2}$, $\qquad$ Then, $\qquad t = \left(K_1{L_1}^2 + K_2{L_2}^2\right)\emptyset$

Torque t is input while angular displacement $\emptyset$ is output,

$$\emptyset = \frac{1}{K_1{L_1}^2 + K_2{L_2}^2} t$$

And the associated block diagram is,

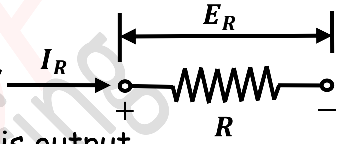


Representation of Electric Components:

Basic electric elements are Resistor, Inductor, and Capacitor:

1) **Electric Resistor**:

Voltage drop across Resistor with R Resistance as shown in **Figure**,



$$E_R = RI_R$$

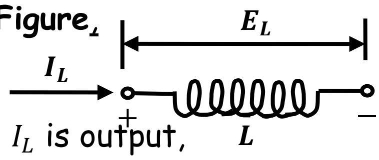
Electrical impedance of Resistor is, $Z_R = R$, also E_R is input and I_R is output,

$$I_R = \frac{1}{R} E_R$$

Block diagram representation of Electric Resistor as shown in **Figure**,

2) **Electric Inductor**:

Voltage drop across Inductor with L Inductance as shown in **Figure**,

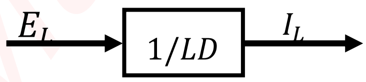


$$E_L = L\frac{dI_L}{dt} = LDI_L$$

Electrical impedance of Inductor is, $Z_L = LD$, also E_L is input and I_L is output,

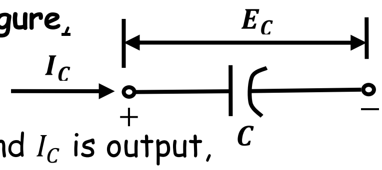
$$I_L = \frac{1}{LD} E_L$$

Block diagram representation of Electric Inductor as shown in **Figure**,



3) **Electric Capacitor**:

Voltage drop across Capacitor with C Capacitance as shown in **Figure**,

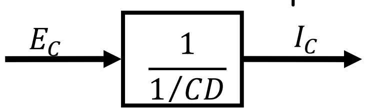


$$E_C = \frac{Q}{C} = \frac{\int I_C dt}{C} = \frac{1}{CD} I_C$$

Electrical impedance of Capacitor is, $Z_C = \frac{1}{CD}$, also E_C is input and I_C is output,

$$I_C = \frac{1}{1/CD} E_C$$

Block diagram representation of Electric Capacitor as shown in **Figure**,

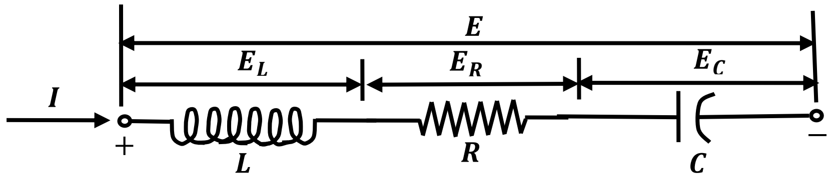


Representation of Electric Components in Series:

Electric laws for elements arranged in series combination are,

1. $E = E_1 + E_2 + E_3 + \cdots$ Total Voltage Drop.
2. $I = I_1 = I_2 = I_3 = \cdots$ Total Current.
3. $Z = Z_1 + Z_2 + Z_3 + \cdots$ Equivalent Electric Impedance.

For example, to obtain mathematical equation of operation for RLC circuit in series combination as shown in **Figure,**



Total Voltage drop across RLC circuit, $E = E_L + E_R + E_C$

$$E = LDI_L + RI_R + \frac{1}{CD}I_C$$

For series combination, $I = I_L = I_R = I_C$, then,

$$E = \left(LD + R + \frac{1}{CD}\right)I$$

Second way, Equivalent Electrical Impedance is,

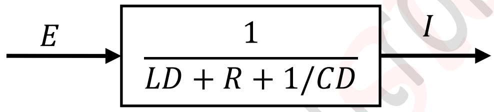
$$Z = Z_L + Z_R + Z_C, \qquad Z = LD + R + \frac{1}{CD}, \qquad Since, \ E = ZI$$

$$E = \left(LD + R + \frac{1}{CD}\right)I$$

Total voltage drop E is input while current I is output,

$$I = \frac{1}{LD + R + 1/CD}E$$

Overall block diagram representation for RLC circuit in series combination shown in **Figure**,



Since, $Q = \int I dt$, and, $I = DQ$, $E = \left(LD^2 + RD + \frac{1}{C}\right)Q$

In which is similar to mathematical mechanical equation in series combination,

$$f = (MD^2 + BD + K)x$$

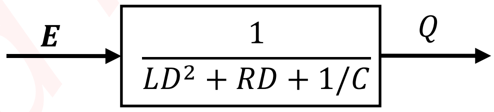
Equivalent Electrical Impedance in this case is,

$$Z = LD^2 + RD + \frac{1}{C}$$

Since total voltage drop E is input and charge Q is output,

$$Q = \frac{1}{LD^2 + RD + 1/C}E$$

Overall block diagram representation for RLC circuit in series combination as in **Figure,**

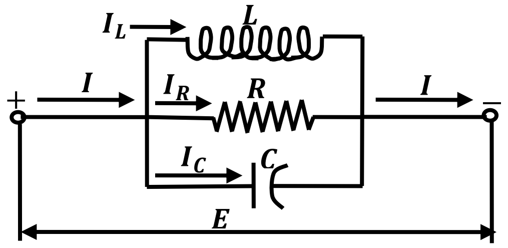


Representation of Electric Components in Parallel:

Electrical laws for elements arranged in parallel combination are,

1. $E = E_1 = E_2 = E_3 = \cdots$ Voltage Drops.
2. $I = I_1 + I_2 + I_3 + \cdots$ Total Current.
3. $Z = \frac{1}{1/Z_1 + 1/Z_2 + 1/Z_3 + \cdots}$ Equivalent Electric Impedance.

For example, to obtain mathematical equation of operation for RLC circuit in parallel combination as shown in **Figure,**



Total current flows across RLC circuit, $I = I_L + I_R + I_C$

For electrical parallel combination, $E = E_L = E_R = E_C$ Then,

$$I = \left(\frac{1}{LD} + \frac{1}{R} + \frac{1}{1/CD}\right)E \qquad E = \frac{1}{\frac{1}{LD} + \frac{1}{R} + \frac{1}{1/CD}} I$$

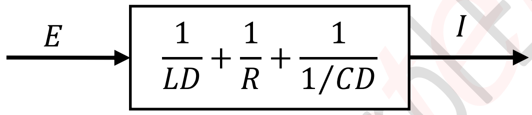
Second way, Electrical Impedance in parallel is,

$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}, \qquad Z = \frac{1}{\frac{1}{LD} + \frac{1}{R} + \frac{1}{1/CD}}, \qquad E = ZI \qquad E = \frac{1}{\frac{1}{LD} + \frac{1}{R} + \frac{1}{1/CD}} I$$

Total voltage drop E is input and current I is output,

$$I = \frac{1}{Z}E \qquad I = \left(\frac{1}{LD} + \frac{1}{R} + \frac{1}{1/CD}\right)E$$

Overall block diagram representation for RLC circuit in parallel combination as in **Figure,**



Since, $Q = \int I dt$, and, $I = DQ$

$$E = \frac{1}{\frac{1}{LD^2} + \frac{1}{RD} + \frac{1}{1/C}} Q$$

In which is similar to mathematical mechanical equation in parallel combination,

$$f = \frac{1}{1/MD^2 + 1/BD + 1/K} x$$

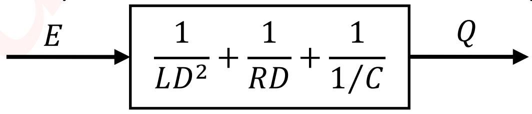
Equivalent Electrical Impedance in this case is,

$$Z = \frac{1}{\frac{1}{LD^2} + \frac{1}{RD} + \frac{1}{1/C}}$$

Since total voltage drop E is input and charge Q is output,

$$Q = \frac{1}{Z}E \qquad Q = \left(\frac{1}{LD^2} + \frac{1}{RD} + \frac{1}{1/C}\right)E$$

Overall block diagram representation for RLC circuit in parallel combination as in **Figure,**



Analogies:

Analogies refer to similarity for different systems like mechanical and electrical in which their equations of operation have same form as in Table.

System	Series Combination	Parallel Combination
Translational Mechanical System	$f=(MD^2+BD+K)x$ $f=(MD+B+K/D)\dot{x}$	$f=\frac{1}{1/MD^2+1/BD+1/K}x$ $f=\frac{1}{1/MD+1/B+1/(K/D)}\dot{x}$
Electrical System	$E=\left(LD^2+RD+\frac{1}{C}\right)Q$ $E=\left(LD+R+\frac{1}{CD}\right)I$	$E=\frac{1}{\frac{1}{LD^2}+\frac{1}{RD}+\frac{1}{1/C}}Q$ $E=\frac{1}{\frac{1}{LD}+\frac{1}{R}+\frac{1}{1/CD}}I$

Analogies are constructed by replacing quantities of one system by another which called analogous quantities. Analogies are classified into two categories:

1) **Direct Analog:** In direct analog, series mechanical elements are replaced by corresponding series electrical elements or parallel mechanical elements are replaced by corresponding electrical elements in parallel and vice-versa. A **Direct Force-Voltage Analog** is constructed as in **Table 1.**

Table 1: Analogous Quantities in a Direct (Force-Voltage) Analog.

Translational Mechanical System	Force f	Velocity $\dot{x}=Dx$	Displacement $\boldsymbol{x}$	Mass M	Viscous Damping Coefficient B	Spring Constant K
Electrical System	Voltage E	Current $I=DQ$	Charge Q	Inductance L	Resistance R	Reciprocal of Capacitance $1/C$

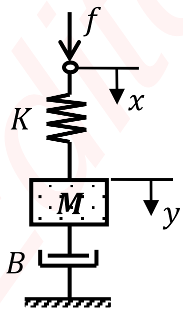
2. **Inverse Analog:** In inverse analog, series mechanical elements are replaced by parallel electrical or parallel mechanical elements are replaced by series electric elements and vice-versa. An **Inverse Force-Current Analog** is constructed as in **Table 2.**

Table 2: Analogous Quantities in an Inverse (Force-Current) Analog.

Translational Mechanical System	Force f	Velocity $\dot{x}$	Displacement x	Mass M	Viscous Damping Coefficient B	Spring Constant K
Electrical System	Current $\boldsymbol{I}$	Voltage E	Integral of Voltage E/D	Capacitance C	Reciprocal of Resistance $1/R$	Reciprocal of Inductance $1/L$

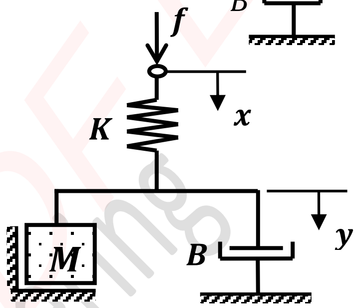
Example 6: Let it be desired to determine the electrical analog for mechanical system shown in **Figure** using:

a. Direct (Force-Voltage) Analog.
b. Inverse (Force-Current) Analog.



Solution:

Equivalent Grounded-Chair Representation can be shown in **Figure**,



Grounded-Chair Representation

Since, $Z_1 = K$, $Z_2 = MD^2 + BD$, total mechanical impedance is,

$$Z = \frac{1}{\frac{1}{Z_1}+\frac{1}{Z_2}} = \frac{1}{1/K + 1/(MD^2+BD)} \qquad f = Zx$$

Differential equation of operation of mechanical system,

$$f = \frac{1}{1/K + 1/(MD^2+BD)}x \qquad f = \frac{K(MD^2+BD)}{MD^2+BD+K}x$$

Second way,

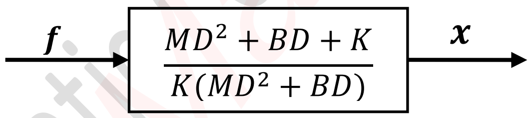
$$x = (x-y)+y \qquad x = \left(\frac{f}{K} + \frac{f}{MD^2+BD}\right) = \left(\frac{1}{K}+\frac{1}{MD^2+BD}\right)f$$

$$f = \frac{1}{1/K + 1/(MD^2+BD)}x = \frac{K(MD^2+BD)}{MD^2+BD+K}x$$

Since f is input and x is output,

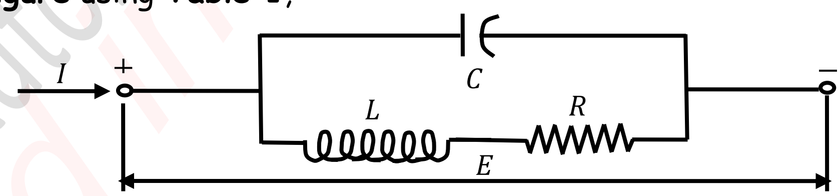
$$x = \frac{1}{Z}f \qquad x = \frac{MD^2+BD+K}{K(MD^2+BD)}f$$

Overall block diagram representation for translational mechanical system is:



a) **Direct (Force-Voltage) Analog:**

Equivalent electrical system of Direct Force-Voltage Analog is represented as shown in **Figure** using **Table 1**,



$$f = \frac{1}{1/K + 1/(MD^2+BD)}x = \frac{K(MD^2+BD)}{MD^2+BD+K}x = \frac{K(MD+B)}{MD^2+BD+K}\dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table1**,

$$E = \frac{1}{1/(1/C) + 1/(LD^2+RD)}Q = \frac{(1/C)\,(LD^2+RD)}{LD^2+RD+1/C}Q = \frac{(1/C)\,(LD+R)}{LD^2+RD+1/C}I$$

Or, from equivalent electrical circuit,

$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{1}{1/(1/CD) + 1/(LD + R)}$$

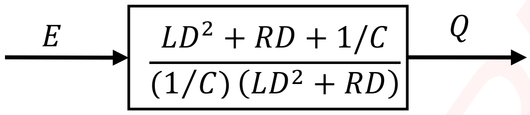
$$E = ZI = \frac{1}{1/(1/CD) + 1/(LD + R)} I \qquad\qquad I = DQ$$

$$E = \frac{1}{1/(1/C) + 1/(LD^2 + RD)} Q = \frac{(1/C)\,(LD^2 + RD)}{LD^2 + RD + 1/C} Q$$

Since E is input and Q is output, then

$$Q = \frac{LD^2 + RD + 1/C}{(1/C)\,(LD^2 + RD)} E$$

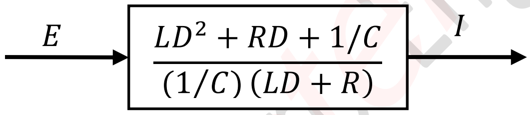
Overall block diagram representation as shown in **Figure**,



Also it could be represented with current I,

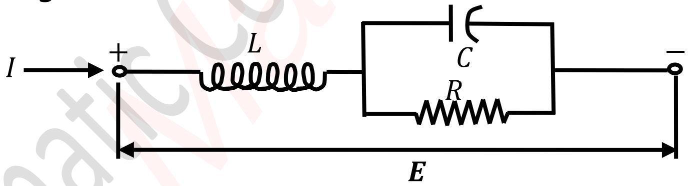
$$E = \frac{(1/C)\,(LD + R)}{LD^2 + RD + 1/C} I \qquad or, \quad I = \frac{LD^2 + RD + 1/C}{(1/C)\,(LD + R)} E$$

And overall block diagram representation as shown in **Figure**,



b) Inverse (Force-Current) Analog:

Equivalent electrical system of inverse force-current analog can be represented as shown in **Figure** using **Table 2**,



Since,

$$f = \frac{1}{1/K + 1/(MD^2 + BD)} x = \frac{K(MD^2 + BD)}{MD^2 + BD + K} x = \frac{K(MD + B)}{MD^2 + BD + K} \dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table 2**,

$$I = \left[\frac{1}{1/(1/L) + 1/(CD^2 + (1/R)D)}\right] \frac{E}{D} = \left[\frac{(1/L)[CD^2 + (1/R)D]}{CD^2 + (1/R)D + (1/L)}\right] \frac{E}{D} = \frac{(1/L)(CD + 1/R)}{CD^2 + (1/R)D + 1/L} E$$

Or, from equivalent electrical circuit,

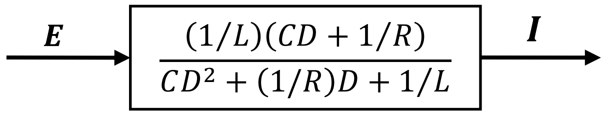
$$Z = Z_1 + Z_2 = LD + \frac{1}{1/(1/CD) + 1/R} = \frac{CD^2 + (1/R)D + 1/L}{(1/L)(CD + 1/R)}$$

$$E = ZI = \left(\frac{CD^2 + (1/R)D + 1/L}{(1/L)(CD + 1/R)}\right) I \qquad\qquad I = \frac{(1/L)(CD + 1/R)}{CD^2 + (1/R)D + 1/L} E$$

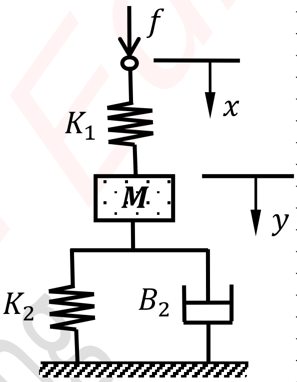
Since E is input and I is output, then

$$I = \frac{(1/L)(CD + 1/R)}{CD^2 + (1/R)D + 1/L}E$$

Overall block diagram representation as shown in **Figure**,



$$\frac{(1/L)(CD + 1/R)}{CD^2 + (1/R)D + 1/L}$$

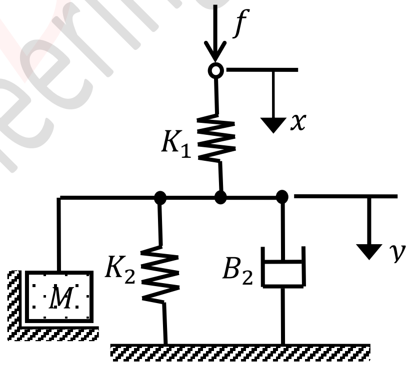


Example 7: Let it be desired to determine the electrical analog for the mechanical system shown in **Figure** using:

a) The direct analog. b)The inverse analog.

Solution:

Equivalent Grounded-Chair Representation can be constructed as shown in **Figure**,



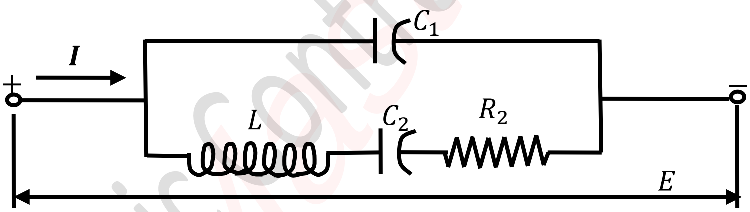
Total impedance is,

$$Z = \frac{1}{\frac{1}{K_1} + \frac{1}{MD^2+B_2D+K_2}} = \frac{K_1(MD^2+B_2D+K_2)}{MD^2+B_2D+K_1+K_2}, \qquad \text{Since,} \quad f = Zx$$

$$f = \frac{K_1(MD^2 + B_2D + K_2)}{MD^2 + B_2D + K_1 + K_2}x = \frac{K_1(MD + B_2 + K_2/D)}{MD^2 + B_2D + K_1 + K_2}\dot{x}$$

a) Direct (Force-Voltage) Analog:

Equivalent electrical system of direct force-voltage analog is represented as shown in **Figure** using **Table 1**,



Since,

$$f = \frac{K_1(MD^2 + B_2D + K_2)}{MD^2 + B_2D + K_1 + K_2}x = \frac{K_1[MD + B_2 + (K_2/D)]}{MD^2 + B_2D + K_1 + K_2}\dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table1**,

$$E = \frac{\frac{1}{C_1}\left(LD^2 + R_2D + \frac{1}{C_2}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}Q = \frac{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}I$$

Or, from the equivalent electrical circuit,

$$Z = \frac{1}{\frac{1}{\frac{1}{C_1D}} + \frac{1}{LD+\frac{1}{C_2D}+R_2}} = \frac{1}{C_1D + \frac{1}{LD+\frac{1}{C_2D}+R_2}} = \frac{LD + \frac{1}{C_2D} + R_2}{C_1D\left(LD + \frac{1}{C_2D} + R_2\right) + 1} = \frac{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}$$

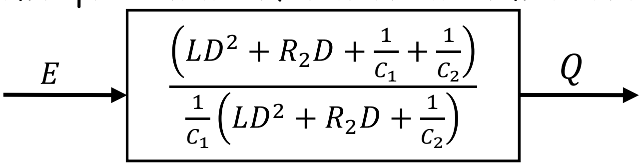
$$Z = \frac{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)} \qquad \text{Since,} \quad E = ZI$$

$$E = \frac{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}I = \frac{\frac{1}{C_1}\left(LD^2 + R_2D + \frac{1}{C_2}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}Q$$

Since E is input and Q is output,

$$Q = \frac{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}{\frac{1}{C_1}\left(LD^2 + R_2D + \frac{1}{C_2}\right)}E$$

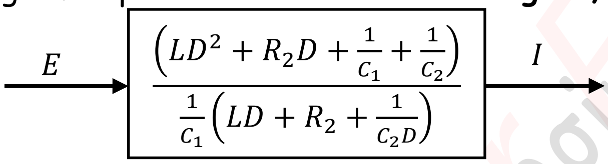
Overall block diagram representation for translation mechanical system is:



Also it could be represented with current I,

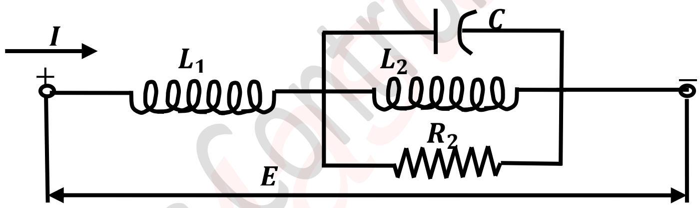
$$E = \frac{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}I \qquad or, \qquad I = \frac{\left(LD^2 + R_2D + \frac{1}{C_1} + \frac{1}{C_2}\right)}{\frac{1}{C_1}\left(LD + R_2 + \frac{1}{C_2D}\right)}E$$

And overall block diagram representation as shown in **Figure**,



b) Inverse (Force-Current) Analog:

Equivalent electrical system of inverse force-current analog can be represented as shown in **Figure** using **Table 2**,



Since,

$$f = \frac{K_1(MD^2 + B_2D + K_2)}{MD^2 + B_2D + K_1 + K_2}x = \frac{K_1[MD + B_2 + (K_2/D)]}{MD^2 + B_2D + K_1 + K_2}\dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table2**,

$$I = \left[\frac{\frac{1}{L_1}\left(CD^2 + \frac{1}{R_2}D + \frac{1}{L_2}\right)}{\left(CD^2 + \frac{1}{R_2}D + \frac{1}{L_1} + \frac{1}{L_2}\right)}\right]\frac{E}{D} = \frac{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2D}\right)}{\left(CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}\right)}E$$

Or, from the equivalent electrical circuit,

$$Z = L_1D + \frac{1}{\frac{1}{\frac{1}{CD}} + \frac{1}{L_2D} + \frac{1}{R_2}} = \frac{CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}}{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2D}\right)}$$

Since $E = ZI$ then,

$$E = \frac{CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}}{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2D}\right)}I \qquad or, \qquad I = \frac{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2D}\right)}{CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}}E$$

Since E is input and I is output,

$$I = \frac{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2 D}\right)}{\left(CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}\right)} E \qquad I = \frac{1}{Z} E$$

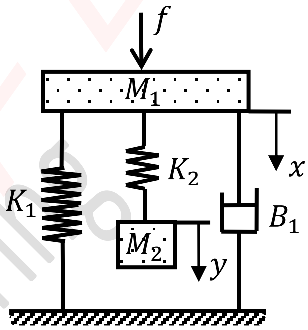
Overall block diagram representation for translation mechanical system is:

$$E \longrightarrow \boxed{\frac{\frac{1}{L_1}\left(CD + \frac{1}{R_2} + \frac{1}{L_2 D}\right)}{CD^2 + \frac{D}{R_2} + \frac{1}{L_1} + \frac{1}{L_2}}} \longrightarrow I$$

Example 8: Let it be desired to determine the electrical analog of mechanical system shown in **Figure** using:

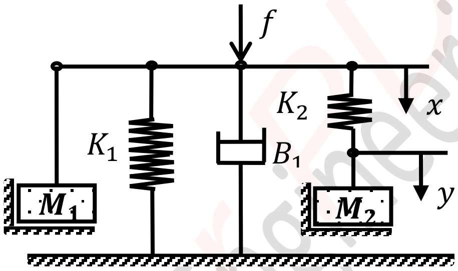
a. Direct (force-voltage) analog.

b. Inverse (force-current) analog.



Solution:

The equivalent grounded chair representation can be constructed as shown in **Figure**. Total impedance,



$$Z = M_1 D^2 + K_1 + B_1 D + \frac{1}{\frac{1}{K_2} + \frac{1}{M_2 D^2}}$$

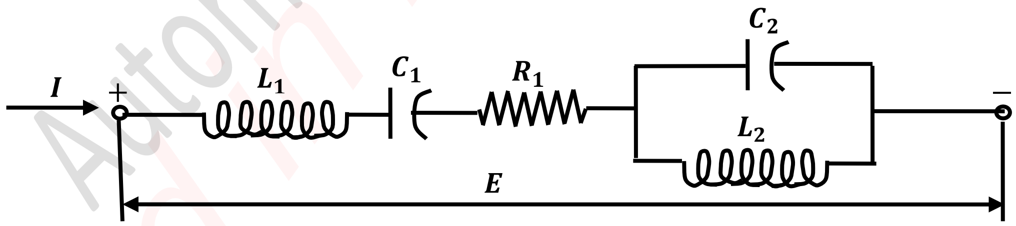
$$= M_1 D^2 + K_1 + B_1 D + \frac{M_2 D^2}{\frac{1}{K_2}(M_2 D^2 + K_2)}$$

Since, $f = Zx$

$$f = \left(M_1 D^2 + B_1 D + K_1 + \frac{M_2 D^2}{\frac{1}{K_2}(M_2 D^2 + K_2)}\right) x = \left(M_1 D + B_1 + \frac{K_1}{D} + \frac{M_2 D}{\frac{1}{K_2}(M_2 D^2 + K_2)}\right) \dot{x}$$

a) Direct (force-voltage) analog,

Equivalent electrical of direct force-voltage analog as shown in **Figure** using **Table 1**,



Since,

$$f = \left(M_1 D^2 + B_1 D + K_1 + \frac{M_2 D^2}{\frac{1}{K_2}(M_2 D^2 + K_2)}\right) x = \left(M_1 D + B_1 + \frac{K_1}{D} + \frac{M_2 D}{\frac{1}{K_2}(M_2 D^2 + K_2)}\right) \dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table1**,

$$E = \left[L_1 D^2 + R_1 D + \frac{1}{C_1} + \frac{\frac{1}{C_2} L_2 D^2}{L_2 D^2 + \frac{1}{C_2}}\right] Q = \left[L_1 D + R_1 + \frac{1}{C_1 D} + \frac{\frac{1}{C_2} L_2 D}{L_2 D^2 + \frac{1}{C_2}}\right] I$$

Or, from the equivalent electrical circuit,

$$Z = L_1D + \frac{1}{C_1D} + R_1 + \frac{1}{\frac{1}{\frac{1}{C_2D}} + \frac{1}{L_2D}} = L_1D + \frac{1}{C_1D} + R_1 + \frac{1}{C_2D + \frac{1}{L_2D}} = L_1D + \frac{1}{C_1D} + R_1 + \frac{L_2D}{C_2L_2D^2 + 1}$$

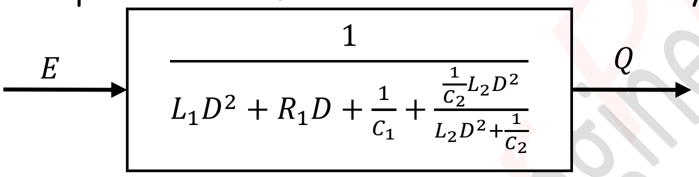
$$Z = L_1D + \frac{1}{C_1D} + R_1 + \frac{\frac{1}{C_2}L_2D}{L_2D^2 + \frac{1}{C_2}} \qquad Since,\ \ E = ZI$$

$$E = \left[L_1D + R_1 + \frac{1}{C_1D} + \frac{\frac{1}{C_2}L_2D}{L_2D^2 + \frac{1}{C_2}}\right]I = \left[L_1D^2 + R_1D + \frac{1}{C_1} + \frac{\frac{1}{C_2}L_2D^2}{L_2D^2 + \frac{1}{C_2}}\right]Q$$

Since E is input and Q is output,

$$Q = \frac{1}{L_1D^2 + R_1D + \frac{1}{C_1} + \frac{\frac{1}{C_2}L_2D^2}{L_2D^2 + \frac{1}{C_2}}}E$$

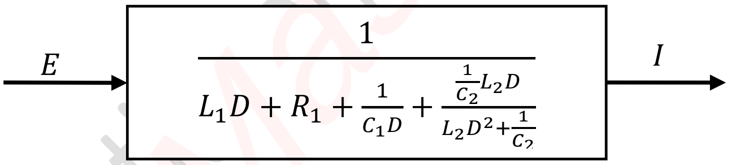
Overall block diagram representation for translation mechanical system is:



Also it could be represented with current I,

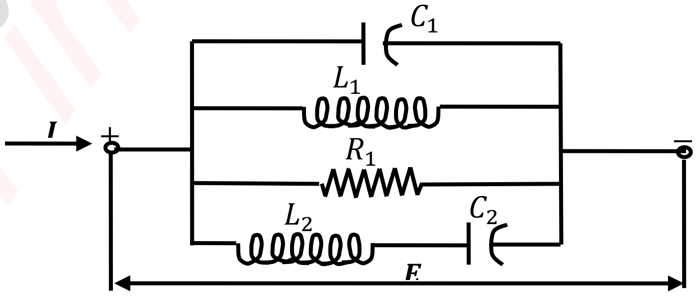
$$E = \left[L_1D + R_1 + \frac{1}{C_1D} + \frac{\frac{1}{C_2}L_2D}{L_2D^2 + \frac{1}{C_2}}\right]I \qquad or, \quad I = \frac{1}{L_1D + R_1 + \frac{1}{C_1D} + \frac{\frac{1}{C_2}L_2D}{L_2D^2 + \frac{1}{C_2}}}E$$

And overall block diagram representation as shown in **Figure**,



b) Inverse (force-current) analog,

Equivalent electrical system of inverse force-current analog for mechanical system is represented as shown in **Figure**, using **Table2**,



Since,

$$f = \left(M_1D^2 + B_1D + K_1 + \frac{K_2M_2D^2}{M_2D^2 + K_2}\right)x = \left(M_1D + B_1 + \frac{K_1}{D} + \frac{K_2M_2D}{M_2D^2 + K_2}\right)\dot{x}$$

Equivalent electrical differential equation of mechanical system using **Table 2**,

$$I=\left[C_1D^2+\frac{1}{R_1}D+\frac{1}{L_1}+\frac{\frac{1}{L_2}C_2D^2}{C_2D^2+\frac{1}{L_2}}\right]\frac{E}{D}=\left[C_1D+\frac{1}{R_1}+\frac{1}{L_1D}+\frac{\frac{1}{L_2}C_2D}{C_2D^2+\frac{1}{L_2}}\right]E$$

Or, from the equivalent electrical circuit,

$$Z=\frac{1}{\frac{1}{\frac{1}{C_1D}}+\frac{1}{L_1D}+\frac{1}{R_1}+\frac{1}{L_2D+\frac{1}{C_2D}}}=\frac{1}{C_1D+\frac{1}{L_1D}+\frac{1}{R_1}+\frac{C_2D}{L_2C_2D^2+1}}=\frac{1}{C_1D+\frac{1}{R_1}+\frac{1}{L_1D}+\frac{\frac{1}{L_2}C_2D}{C_2D^2+\frac{1}{L_2}}}$$

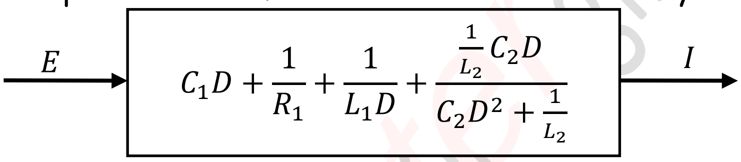
Since, $E=ZI$

$$E=\frac{1}{C_1D+\frac{1}{R_1}+\frac{1}{L_1D}+\frac{\frac{1}{L_2}C_2D}{C_2D^2+\frac{1}{L_2}}}I \qquad I=\left[C_1D+\frac{1}{R_1}+\frac{1}{L_1D}+\frac{\frac{1}{L_2}C_2D}{C_2D^2+\frac{1}{L_2}}\right]E$$

Since E is input and I is output,

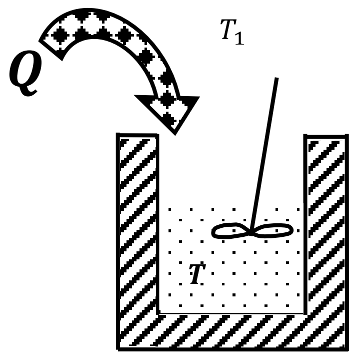
$$I=\left[C_1D+\frac{1}{R_1}+\frac{1}{L_1D}+\frac{\frac{1}{L_2}C_2D}{C_2D^2+\frac{1}{L_2}}\right]E \qquad I=\frac{1}{Z}E$$

Overall block diagram representation for translation mechanical system is:



Representation of Thermal Systems:

Thermal systems are those that involve storage and transfer of heat.

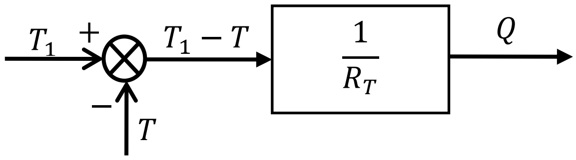


I. Rate of heat transferred into a body is proportional to the difference of temperature across the body,

$$Q\propto(T_1-T)$$

$$Q=hA(T_1-T)=\frac{1}{R_T}(T_1-T)$$

Q =Rate of heat transferred into the body
h =Coefficient of heat transfer of the surface of the body
A =Surface area, T =Temperature of the body
T_1 =Temperature of the surrounding medium
$R_T=1/hA$, Equivalent thermal resistance (Thermal impedance)
Since, T_1-T is input and Q is output, the block diagram representation is,



II. Time rate of change of temperature of the body (dT/dt) is proportional to the rate of heat transferred into the body.

$$Q\propto\frac{dT}{dt} \qquad Q=Mc\frac{dT}{dt}=C_TDT$$

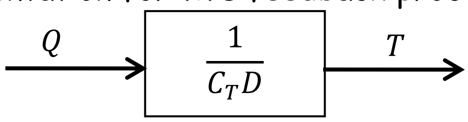
c =Average specific heat of the body
M =Mass of the body

$C_T = M * c$ =Equivalent thermal capacitance

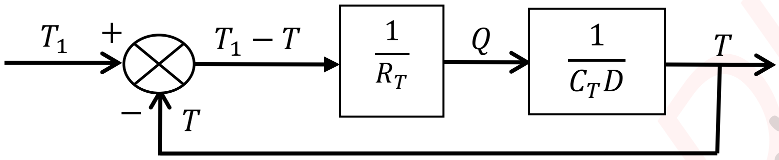
Since, Q is input and T is output then,

$$T = \frac{1}{C_T D} Q$$

Block diagram representation for this feedback process as shown in **Figure**,

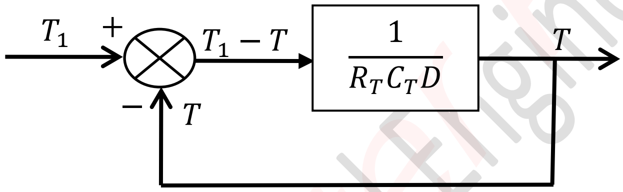


Overall block diagram representation of thermal system is shown in **Figure**,

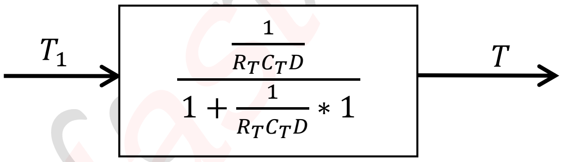


To simplify the above block diagram some steps should be taken into consideration,

1) Using the rule of combining blocks in cascade,



2) Using the rule of eliminating a feedback loop,



And the transfer function or the mathematical differential equation of operation is,

$$\frac{T}{T_1} = \frac{1}{1 + R_T C_T D}$$

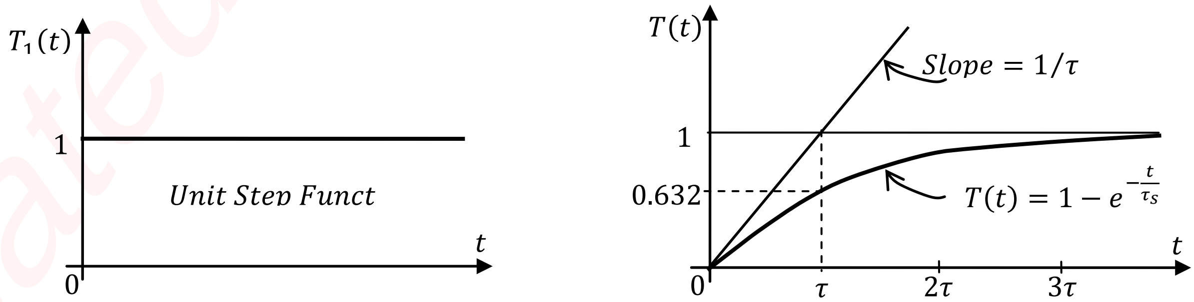
The time constant of the thermal system is, $\tau_c = R_T C_T$,

$$T = \frac{1}{1 + \tau_c D} T_1$$

The general solution or the dynamic response for the first-order differential equation of operation for a unit step input ($T_1 = 1$) is,

$$T(t) = 1 - e^{-\frac{t}{\tau_s}}$$

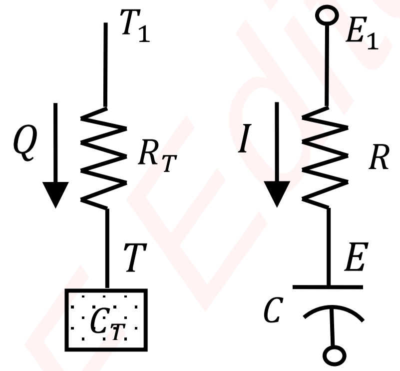
In which the response of the thermal system to a unit step input is shown in the **Figure**,



Analogies of Thermal Systems:

1. Direct Temperature-voltage Analog:

Thermal system may be replaced by an equivalent direct electrical circuit as shown in **Figure,**



$$Q = \frac{1}{R_T}(T_1 - T) = C_T DT \qquad T = \frac{T_1}{1 + R_T C_T D}$$

Equivalent electrical circuit in series, the mathematical differential equation of operation,

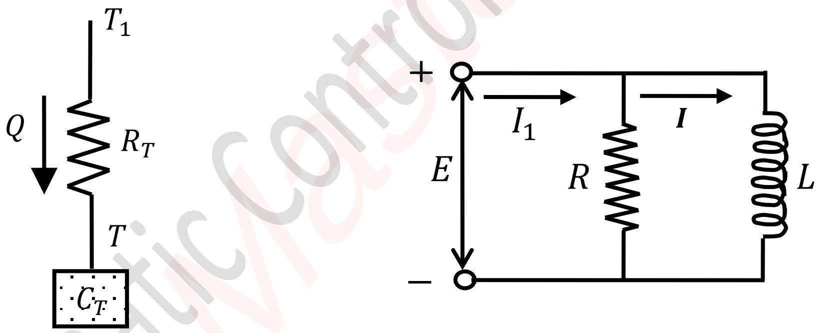
$$I = \frac{1}{R}(E_1 - E) = CDE \qquad E = \frac{E_1}{1 + RCD}$$

Direct Temperature-voltage analog as in the table,

Thermal System	Temperature T	Rate of heat flow Q	Thermal Resistance $\boldsymbol{R_T}$	Thermal Capacitance $\boldsymbol{C_T}$
Electrical System	Voltage E	Current I	Resistance R	Capacitance C

2. Inverse Temperature-current Analog:

Thermal system could be replaced by an equivalent inverse electrical circuit as shown in **Figure,**



For thermal system, the mathematical differential equation of operation is:

$$Q = \frac{1}{R_T}(T_1 - T) = C_T DT \qquad T = \frac{T_1}{1 + R_T C_T D}$$

And equivalent electrical circuit in parallel,

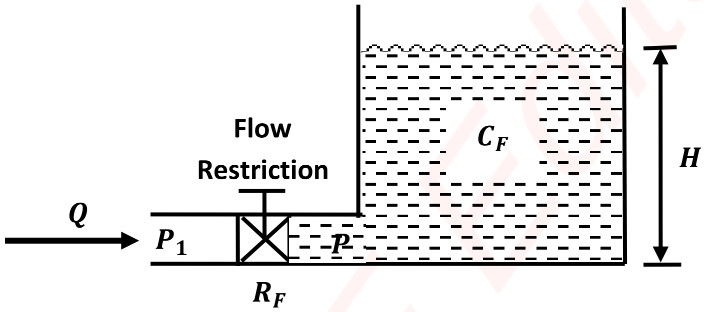
$$E = R(I_1 - I) = LDI \qquad I = \frac{I_1}{1 + (L/R)D}$$

Comparison of these equations yields an inverse temperature-current analog as in table.

Thermal System	Temperature T	Rate of heat flow Q	Thermal Resistance $\boldsymbol{R_T}$	Thermal Capacitance $\boldsymbol{C_T}$
Electrical System	Current I	Voltage E	Reciprocal of Resistance $1/R$	inductance L

Representation of Fluid Systems (Incompressible Fluid):

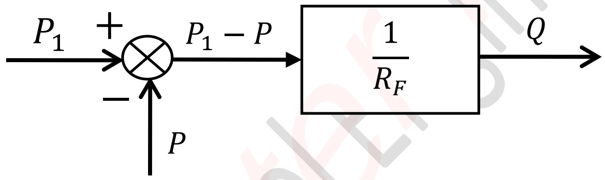
For incompressible fluid, density is constant with variable tank volume. The **Figure** shows an incompressible fluid system in which the external pressure (P_1) is the input while the height of fluid into tank (H) is the output. To present the block diagram representation for the incompressible fluid system,



1. Volume rate of fluid flow (Q) is proportional to pressure drop.

$$Q = \frac{1}{R_F}(P_1 - P)$$

Where, R_F is the Equivalent Resistance of Fluid flow. Since, $P_1 - P$ is input and Q is output, the block diagram representation for this process is,



2. Time rate of change of height of the fluid (dH/dt) into a tank is proportional to the volume rate of flow into the tank.

$$Q \propto \frac{dH}{dt} \qquad Q = A\frac{dH}{dt}$$

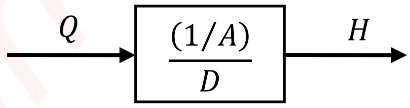
Volume Rate of flow into a tank is,

$$Q = ADH$$

Where Q is input and the height of fluid H is output,

$$H = \frac{(1/A)}{D}Q$$

Block diagram represents for this process,



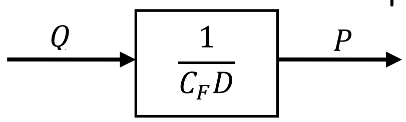
3. Automatic Feedback Flow,

$$Q = ADH = \frac{A}{\rho g}DP = C_F DP \qquad C_F = \frac{A}{\rho g}$$

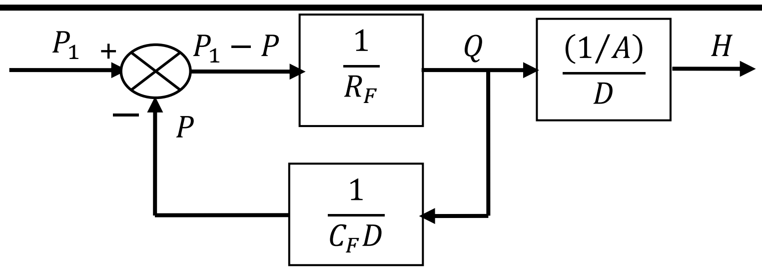
Where, C_F is the Equivalent Fluid Capacitance. Since, Q is input and P is output,

$$P = \frac{1}{C_F D}Q$$

Block diagram represents automatic feedback process,

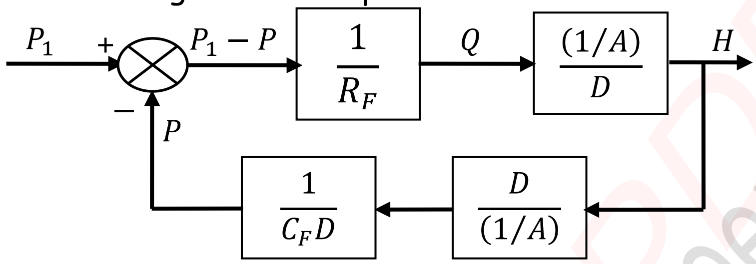


Overall block diagram representation of incompressible fluid flow system,

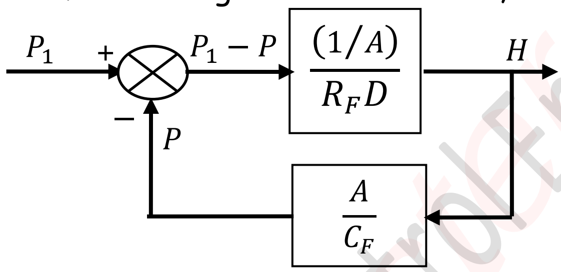


To simplify the above block diagram some steps should be taken into consideration,

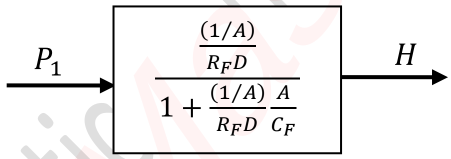
1) Using the rule of moving a take-off point a head of an element,



2) Using the rule of combining blocks in cascade,



3) Using the rule of eliminating a feedback loop,



And the transfer function or the mathematical differential equation of operation is,

$$G(D) = \frac{H}{P_1} = \frac{(C_F/A)}{1 + R_F C_F D}$$

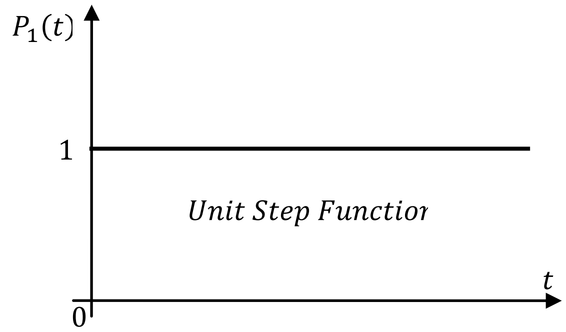
The time constant of the incompressible fluid system is, $\tau_s = R_F C_F$, also the steady-state gain is, $K = (C_F/A)$, yields the first order differential equation,

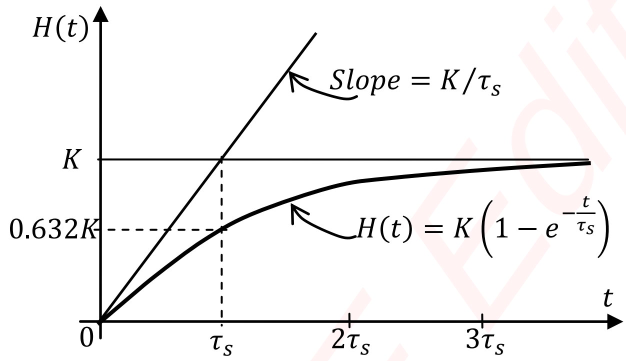
$$H = \frac{K}{1 + \tau_s D} P_1$$

The general solution or the dynamic response for the first-order differential equation of operation for a unit step input ($P_1 = 1$) is,

$$H(t) = K\left(1 - e^{-\frac{t}{\tau_s}}\right)$$

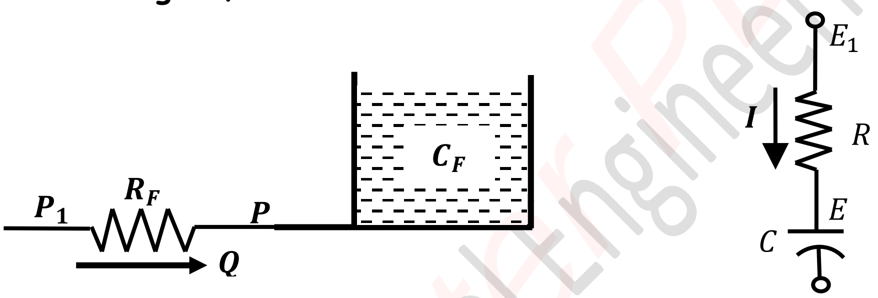
In which the response of the system to a unit step input could be shown in the **Figure**,





<u>**Analogies of Incompressible Fluid Systems:**</u>

1. Direct Pressure-voltage Analog:
   Incompressible fluid system could be replaced by an equivalent direct electrical circuit as shown in **Figure**,



For fluid system, the mathematical differential equation of operation is:

$$Q = \frac{1}{R_F}(P_1 - P) = C_F DP \qquad\qquad P = \frac{P_1}{1 + R_F C_F D}$$

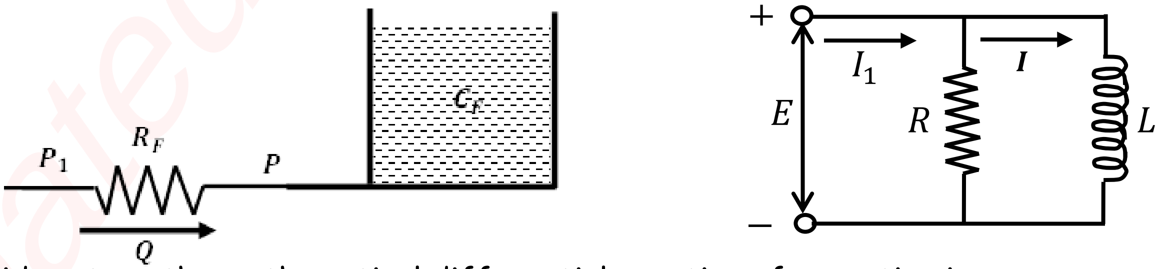
For equivalent electrical circuit in series,

$$I = \frac{1}{R}(E_1 - E) = CDE \qquad\qquad E = \frac{E_1}{1 + RCD}$$

Comparison of these equations gives a Direct Pressure-voltage Analog as in the table.

Fluid System	Pressure P	Volume Rate of flow Q	Fluid Resistance $\boldsymbol{R_F}$	Fluid Capacitance $\boldsymbol{C_F}$
Electrical System	Voltage E	Current I	Resistance R	Capacitance C

2. Inverse pressure-current Analog:
   Fluid system could be replaced by an equivalent inverse electrical circuit as in **Figure**,



For Fluid system, the mathematical differential equation of operation is:

$$Q = \frac{1}{R_F}(P_1 - P) = C_F DP \qquad\qquad P = \frac{P_1}{1 + R_F C_F D}$$

Mathematical differential equation of operation of parallel electrical circuit is:

$$E = R(I_1 - I) = LDI \qquad\qquad I = \frac{I_1}{1 + (L/R)D}$$

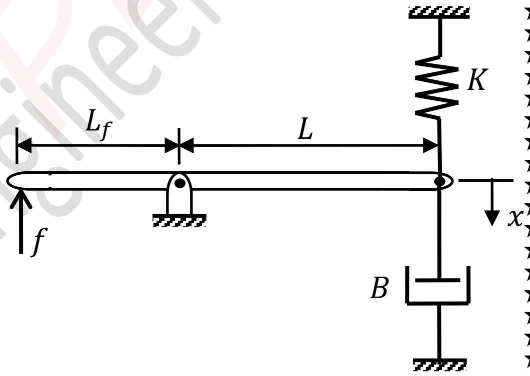
Comparison of these equations gives an inverse pressure-current analog as in the table.

Fluid System	Pressure P	Volume Rate of flow Q	Equivalent Fluid Resistance $\boldsymbol{R_F}$	Equivalent Fluid Capacitance $\boldsymbol{C_F}$
Electrical System	Current I	Voltage E	Reciprocal of Resistance $1/R$	inductance L

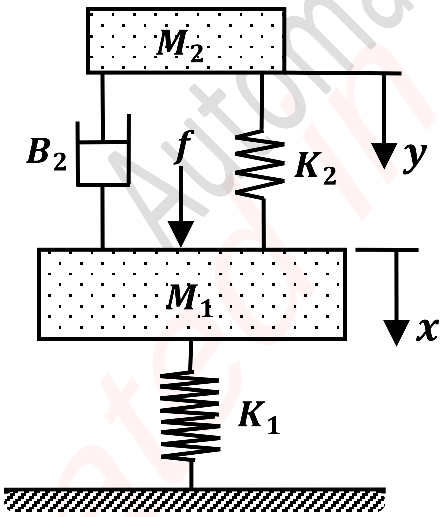
Home Work:

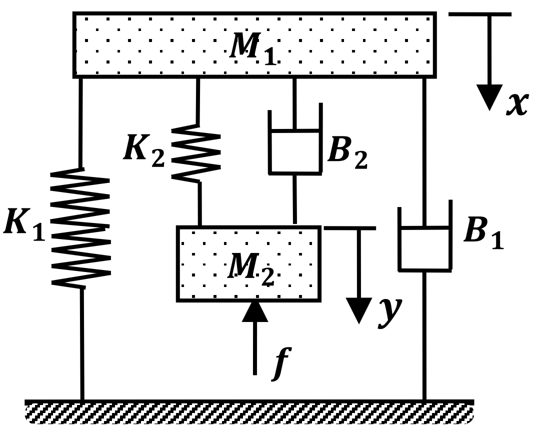
Q1: The lever system shown in **Figure** is drawn in its reference position. The variation in spring position is designated by x and the variation in applied force is designated by f (and x are zero at the reference position).

1) Determine the equation relating f and x.
2) Determine the relationship between t and θ where $t = fL$ and $x = L\theta$



Q2: For mechanical system shown in **Figure** construct the equivalent Grounded-Chair Representation and determine equations relate f and x, f and y, x and y.Then represent the associated block diagram.





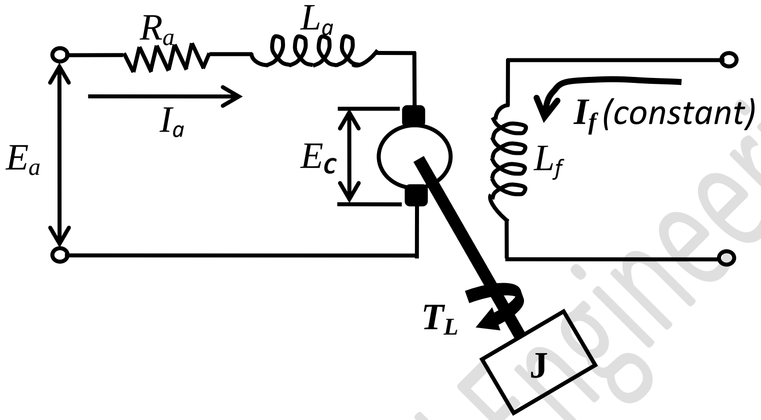
Chapter 3

REPRESENTATION OF CONTROL SYSTEMS

Operating characteristics Determination of control systems are based on information obtained by overall block diagram representation.

I. Armature Controlled DC Servomotors:

In armature controlled DC servomotor shown in **Figure** where voltage E_a is input and $\dot{\theta}$ is output, the field current $\boldsymbol{I_f}$ is constant.



1. DC Motor:

Torque developed by DC motor is,

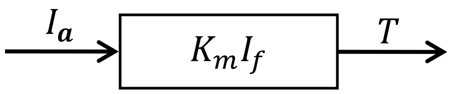
$$T = K_1 \emptyset I_a$$

Where, K_1 is constant of motor and, $\emptyset$ is magnetic flux of the field,

$$\emptyset = K_2 I_f \qquad T = K_1 K_2 I_f I_a$$

let, $K_m = K_1 K_2$ $\qquad T = K_m I_f I_a$

Armature current I_a is input and torque T is output, Block diagram shown in **Figure**:



2. Torque balance:

Torque balance of output shaft DC motor,

$$T - T_L - T_d - T_s = JD^2\theta \qquad T - T_L = (JD^2 + B_v D)\theta$$

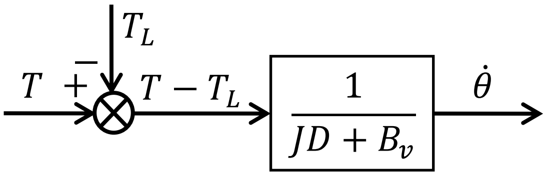
Where T_L is load torque,

$$T - T_L = (JD + B_v)\dot{\theta}$$

Torque difference $(T - T_L)$ is input and angular speed $\dot{\theta}$ is output,

$$\dot{\theta} = \frac{1}{JD + B_v}(T - T_L)$$

Block diagram of torque balance is represented as shown in **Figure**,



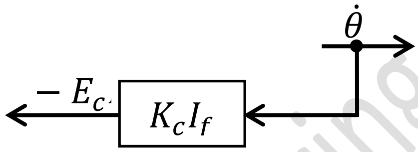
3. Counter Voltage E_c (feedback Process):

Voltage E_c is considered as counter emf induced by rotation of an armature inside magnetic field,

$$E_c = K_3 \emptyset \dot{\theta} \qquad \emptyset = K_2 I_f \qquad E_c = K_2 K_3 I_f \dot{\theta}$$

let, $K_c = K_2 K_3$ $\qquad E_c = K_c I_f \dot{\theta}$

Angular speed $\dot{\boldsymbol{\theta}}$ is input and $\boldsymbol{E_c}$ is output, then block diagram of this feedback process is shown in **Figure**,



4. Electric Circuit:

Electric circuit equation of an armature controlled DC servomotor is,

$$E_a - E_c = (R_a + L_a D) I_a \qquad \text{or,} \qquad E_a - E_c = R_a \left(1 + \frac{L_a}{R_a} D\right) I_a$$

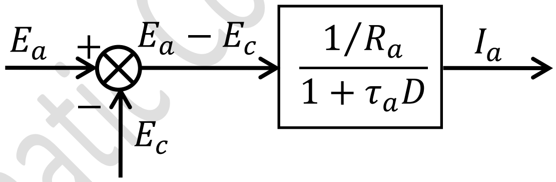
Let, $\boldsymbol{\tau_a} = \frac{\boldsymbol{L_a}}{\boldsymbol{R_a}}$ time constant of armature circuit,

$$E_a - E_c = R_a (1 + \tau_a D) I_a$$

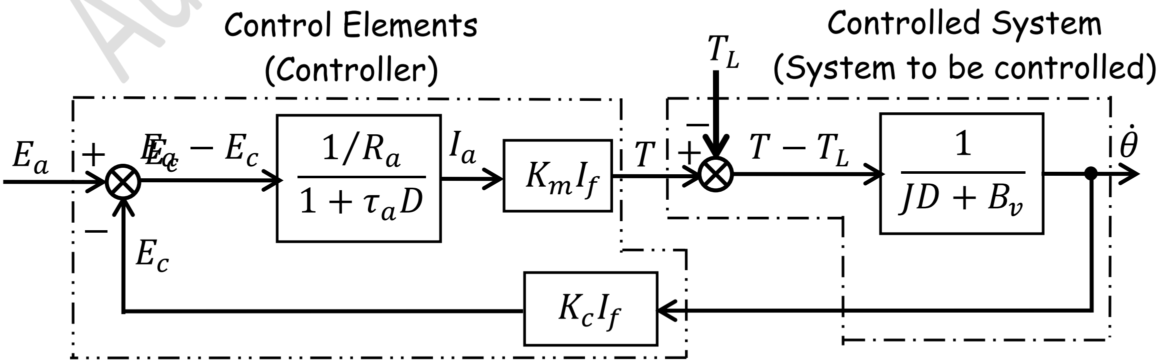
Voltage drop $(E_a - E_c)$ is input and I_a is output,

$$I_a = \frac{1/R_a}{1 + \tau_a D} (E_a - E_c)$$

Block diagram representation of electric circuit is shown in **Figure**,



Overall block diagram representation of an armature controlled DC servomotor is obtained by combining all the processes and components as shown in **Figure**,



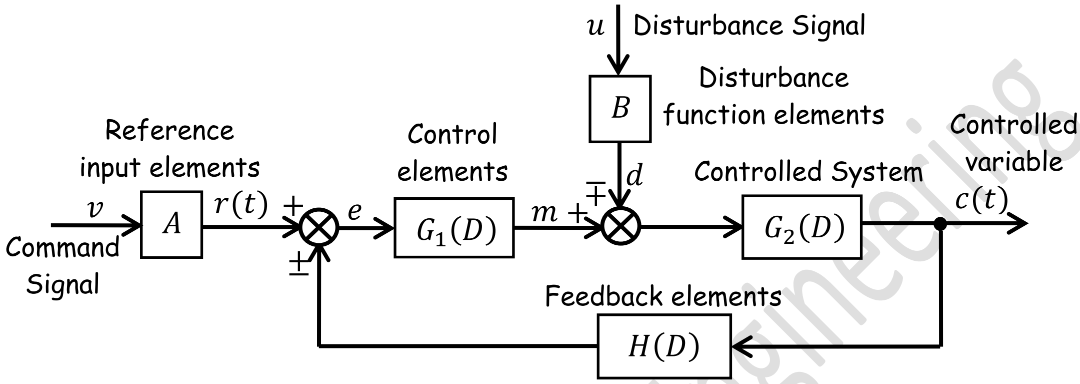
Mathematical differential equation of operation could be derived algebraically from overall block diagram representation,

$$\dot{\theta} = \left[\left(E_a - K_c I_f \dot{\theta}\right)\frac{(1/R_a)}{1+\tau_a D}K_m I_f - T_L\right]\frac{1}{JD+B_v}$$

$$\dot{\theta} = \frac{(1/R_a)K_m I_f E_a - (1+\tau_a D)T_L}{(1+\tau_a D)(JD+B_v) + (1/R_a)\left(K_m I_f\right)\left(K_c I_f\right)}$$

General Block Diagram Representations of a Control System:

1. General block diagram representation of a control system with two inputs and single output (MISO) is shown in **Figure**,



Mathematical equation of operation of general control system with two inputs and one output is,

$$c(t) = \frac{G_1(D)G_2(D)r(t) \mp G_2(D)d(t)}{1 \mp G_1(D)G_2(D)H(D)}$$

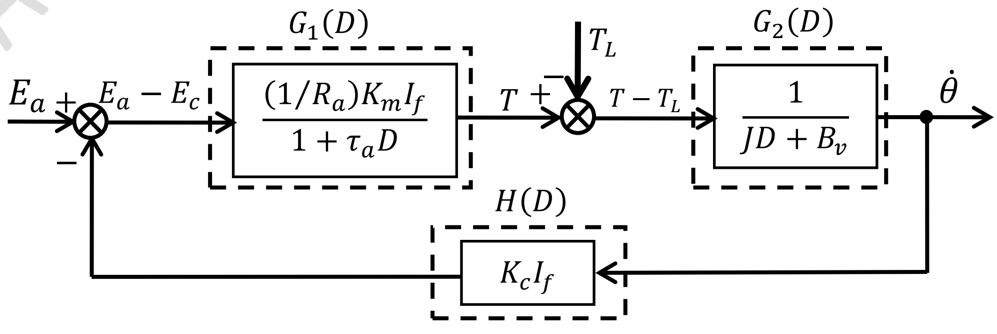
Functions, $G_1(D)$, $G_2(D)$, and $H(D)$ also may be written as:

$$G_1(D) = \frac{N_{G1}}{D_{G1}}, \qquad G_2(D) = \frac{N_{G2}}{D_{G2}}, \qquad H(D) = \frac{N_H}{D_H}$$

Where, N_{G1} is the numerator of G_1 and D_{G1} is denominator of G_1 and so on. General equation of operation is,

$$c(t) = \frac{N_{G1}N_{G2}D_H r(t) \mp N_{G2}D_H D_{G1} d(t)}{N_{G1}N_{G2}N_H \mp D_{G1}D_{G2}D_H}$$

Overall block diagram representation of an armature controlled DC servomotor could be rearranged as general block diagram representation,

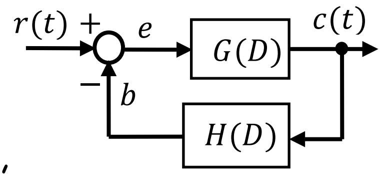


Mathematical differential equation using general equation of operation,

$$c(t) = \frac{N_{G1}N_{G2}D_H r(t) \mp N_{G2}D_H D_{G1} d(t)}{N_{G1}N_{G2}N_H \mp D_{G1}D_{G2}D_H}$$

$$\dot{\theta} = \frac{(1/R_a)K_m I_f E_a - (1+\tau_a D)T_L}{(1+\tau_a D)(JD+B_v) + (1/R_a)(K_m I_f)(K_c I_f)}$$

2. General block diagram representation of a control system with single input and single output (SISO) is shown in **Figure**.



$$c(t) = e * G(D), \quad e = r(t) - b, \quad b = c(t) * H(D),$$

$$c(t) = [r(t) - c(t)H(D)]G(D)$$

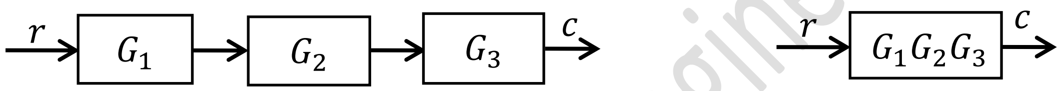
Transfer function of a control system is determined as,

$$\frac{c(t)}{r(t)} = \frac{G(D)}{1 + G(D)H(D)} \qquad \text{Since,} \quad G_D = \frac{N_G}{D_G} \qquad H_D = \frac{N_H}{D_H}$$

$$\frac{c(t)}{r(t)} = \frac{N_G D_H}{N_G N_H + D_G D_H}$$

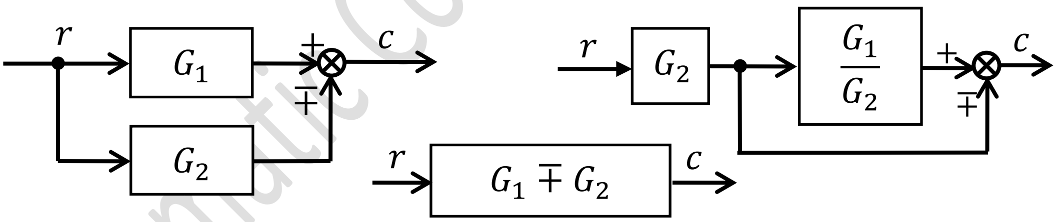
<u>**Rules Block Diagram Reduction:**</u>

1. Combining blocks in cascade

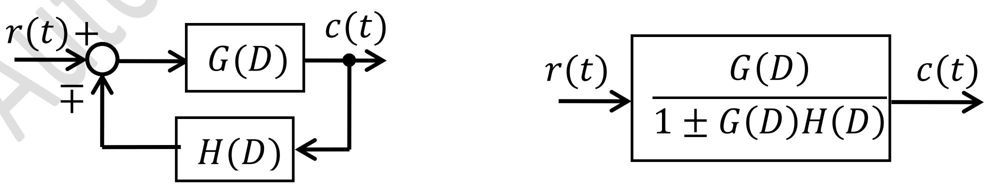


2. Combining blocks in parallel

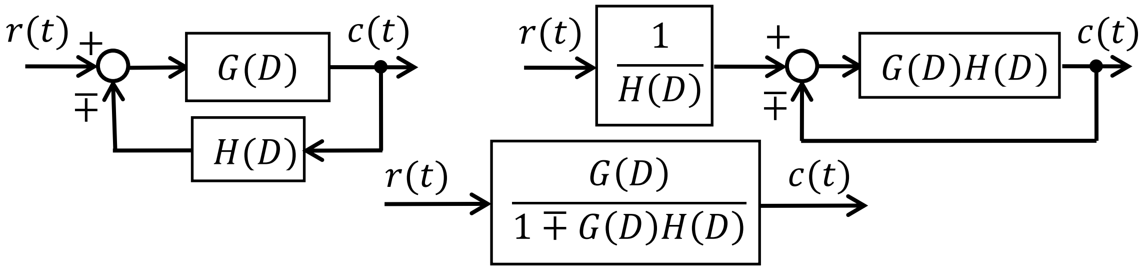
3. Removing a block from a forward path



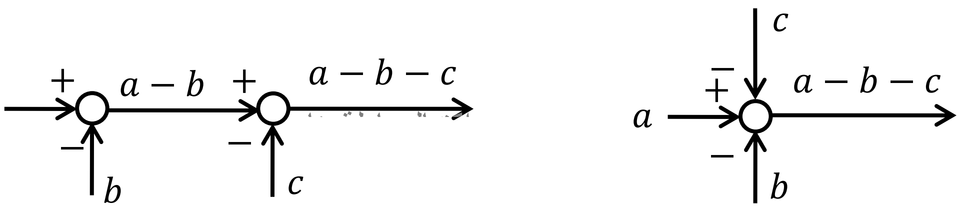
4. Eliminating a feedback loop



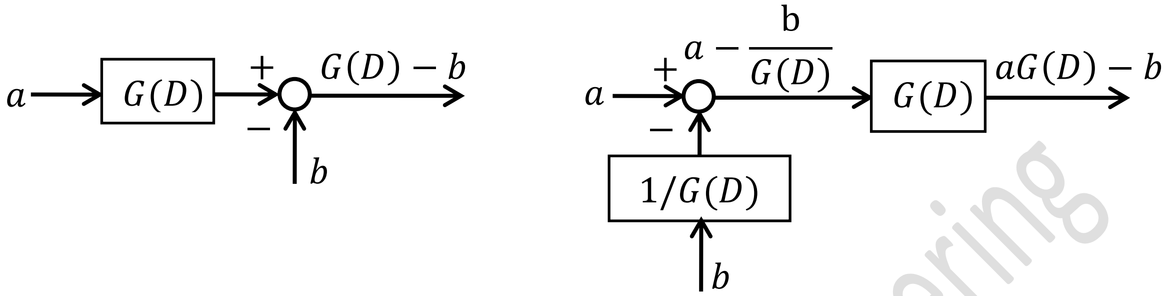
5. Removing a block from a feedback loop



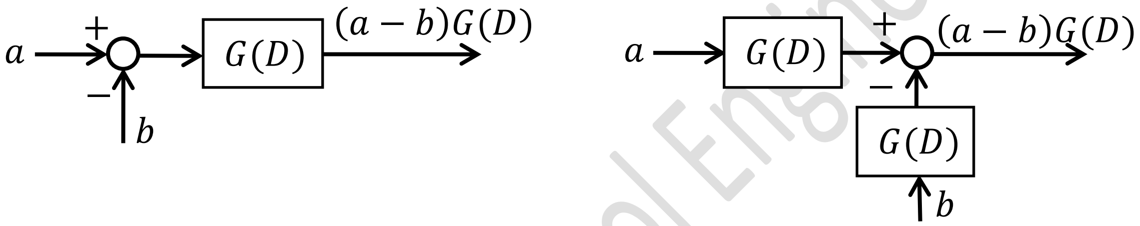
6. Combining interconnected summing points



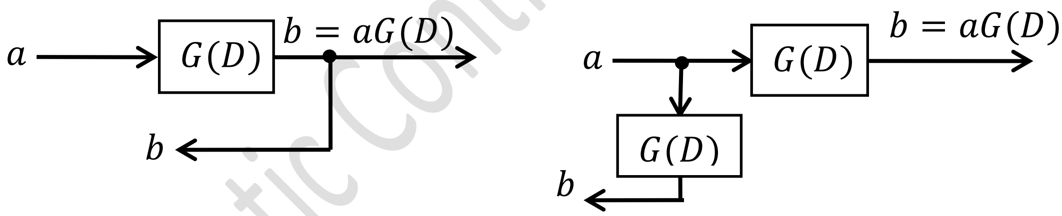
7. Moving a summing point behind an element,



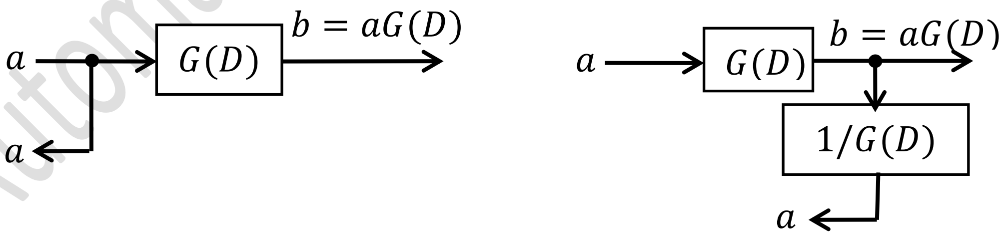
8. Moving a summing point ahead of an element,



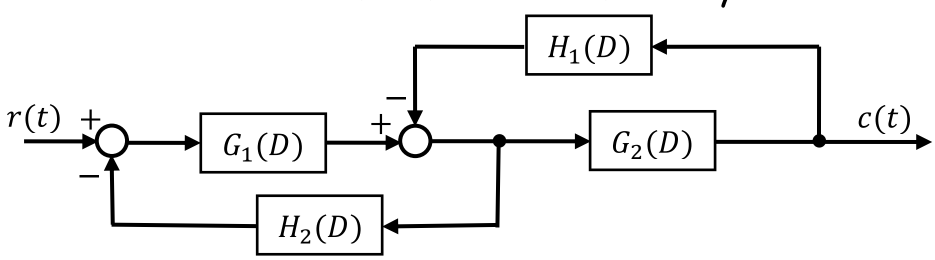
9. Moving a take-off point behind an element,



10. Moving a take-off point a head of an element:

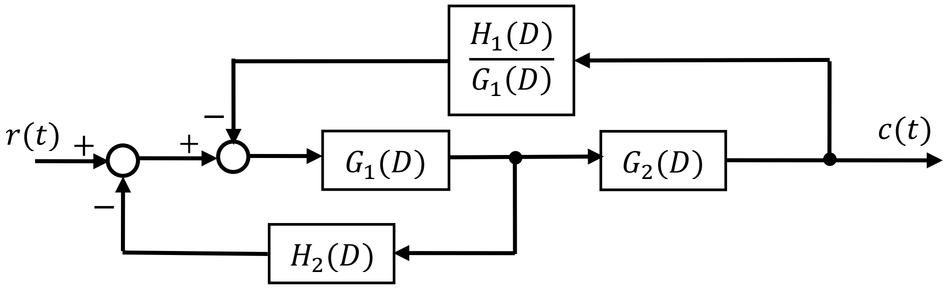


Example 1. Use block-diagram algebra to simplify the block diagram shown in **Figure** and determine the transfer function of the system.

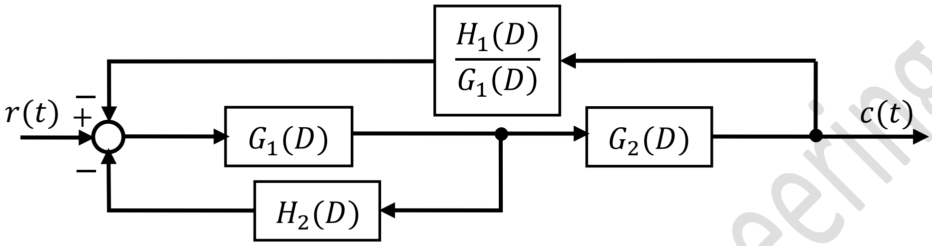


Solution:

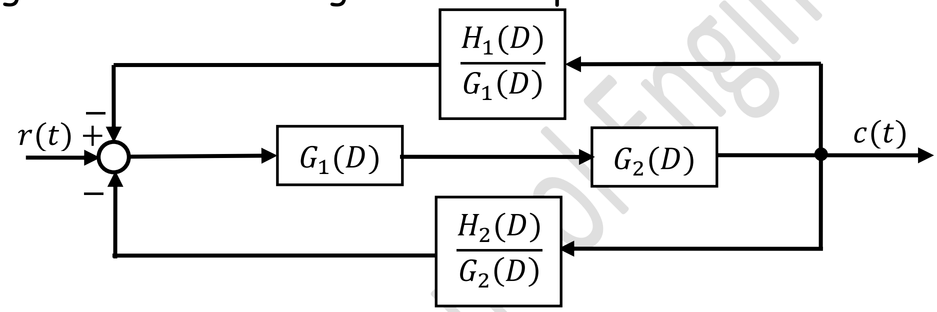
1. Using the rule of moving a summing point behind an element



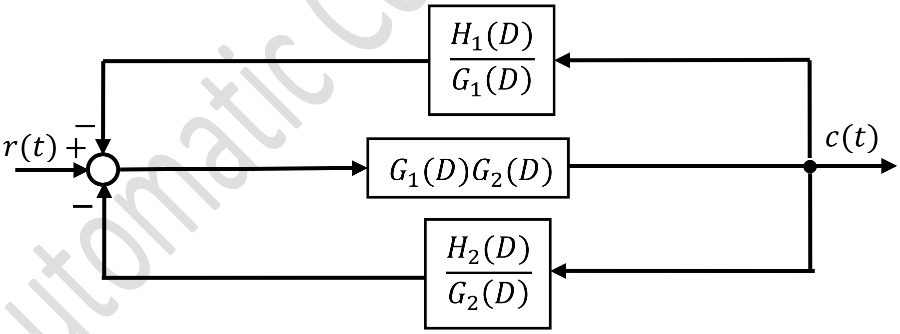
2. Using the rule of combining interconnected summing points



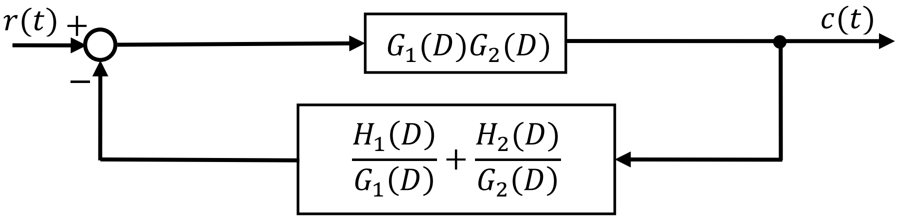
3. Using the rule of moving a take-off point a head of an element



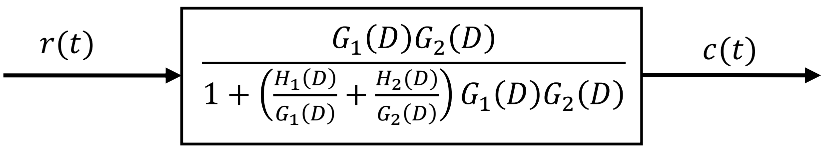
4. Using the rule of combining blocks in cascade



5. Using the rule of combining blocks in parallel



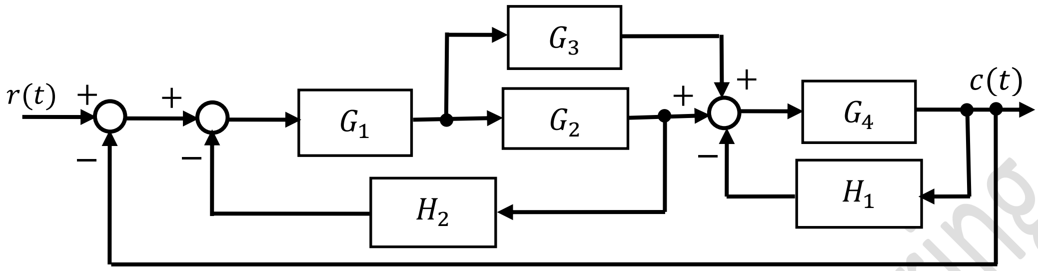
6. Using the rule of eliminating a minor feedback loop



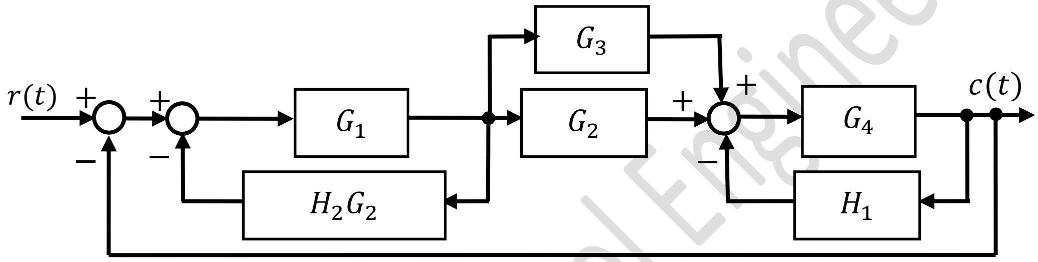
Transfer function of the overall control system,

$$\frac{c(t)}{r(t)} = \frac{G_1(D)G_2(D)}{1+\left(\frac{H_1(D)}{G_1(D)}+\frac{H_2(D)}{G_2(D)}\right)G_1(D)G_2(D)}$$

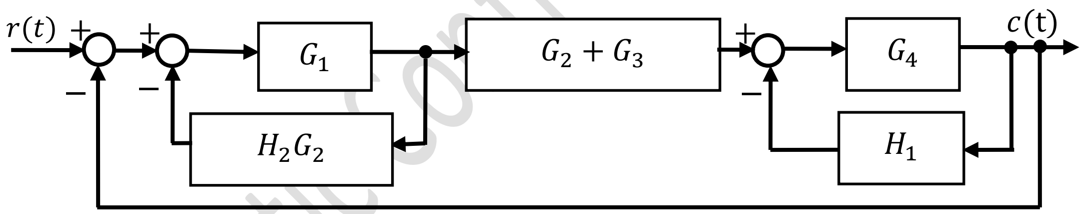
Example 2. Use block-diagram algebra to simplify the block diagram shown in **Figure** and determine the transfer function of the system.



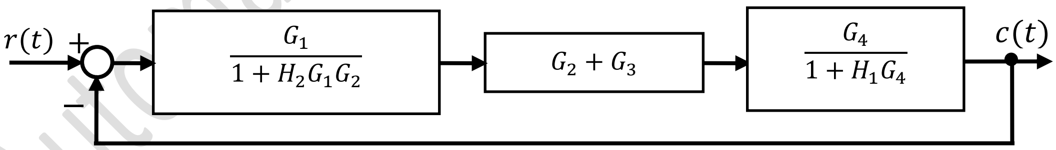
1. Using the rule of moving a take-off point behind an element



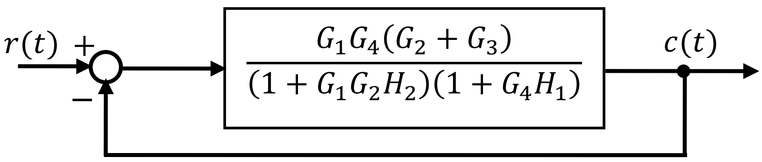
2. Using the rule of combining blocks in parallel



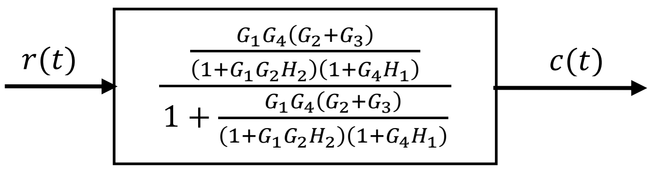
3. Using the rule of eliminating a feedback loop



4. Using the rule of combining blocks in cascade

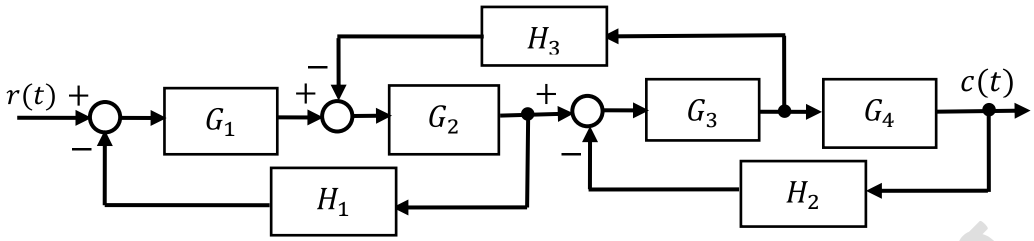


5. Using the rule of eliminating feedback loop

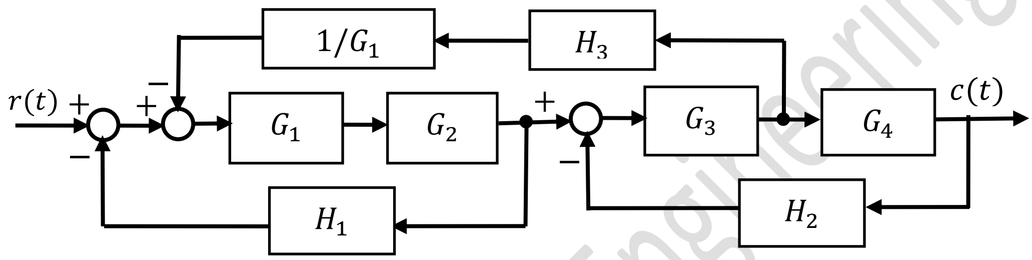


$$\frac{c(t)}{r(t)} = \frac{G_1G_4(G_2+G_3)}{(1+G_1G_2H_2)(1+G_4H_1)+G_1G_4(G_2+G_3)}$$ Transfer Function

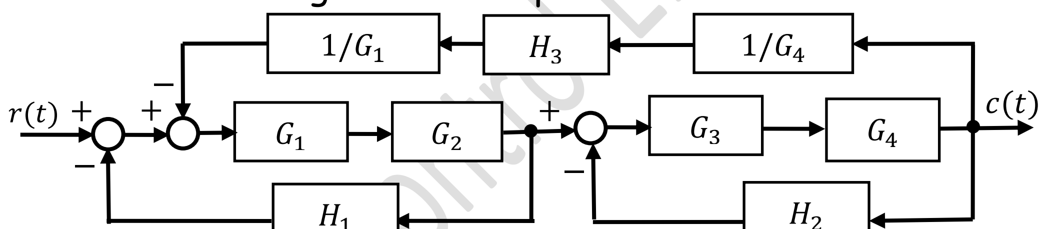
Example 3. Use rules of block-diagram algebra to simplify the block diagram shown in **Figure** and determine the closed loop transfer function.



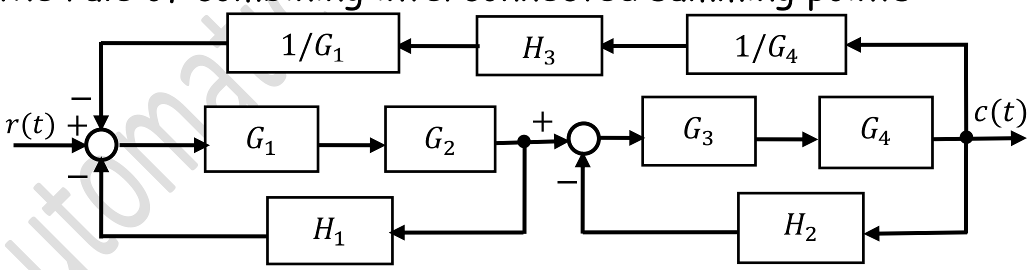
1. Using the rule of moving a summing point behind an element



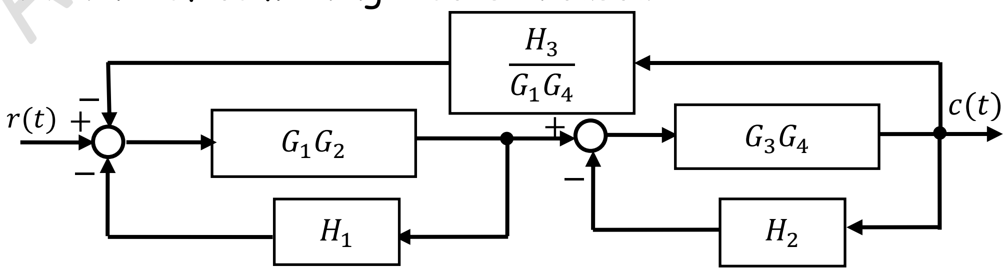
2. Using the rule of moving a take-off point a head of an element



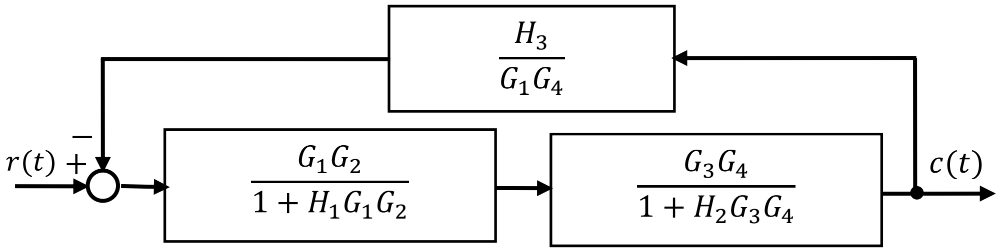
3. Using the rule of combining interconnected summing points



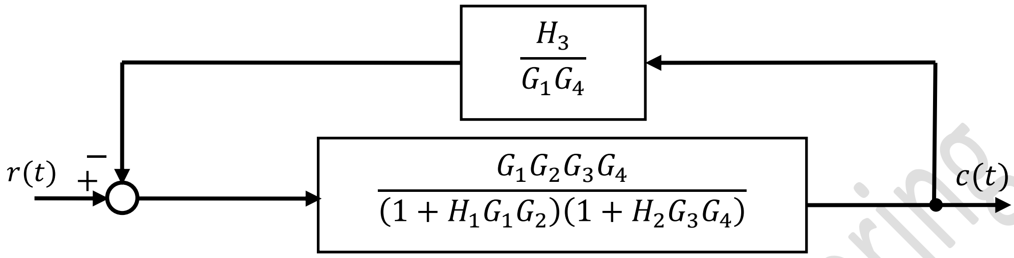
4. Using the rule of combining blocks in cascade



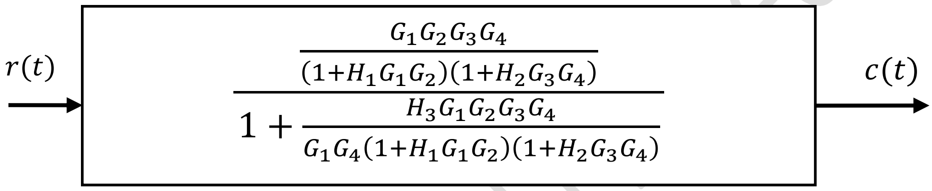
5. Using the rule of eliminating a feedback loop



6. Using the role of combining blocks in cascade



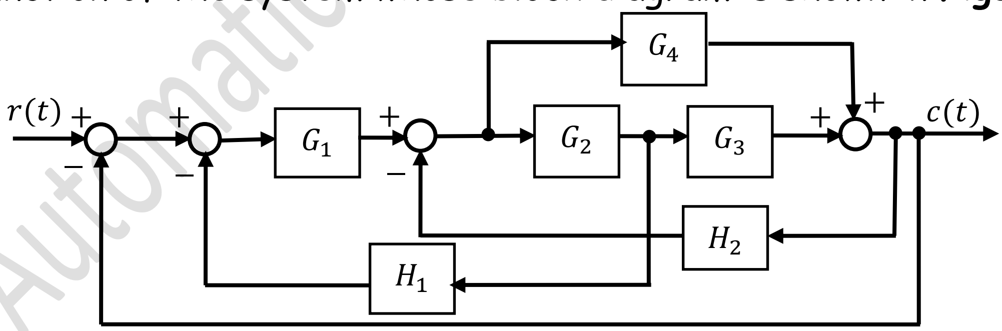
7. Using the rule of eliminating a feedback loop



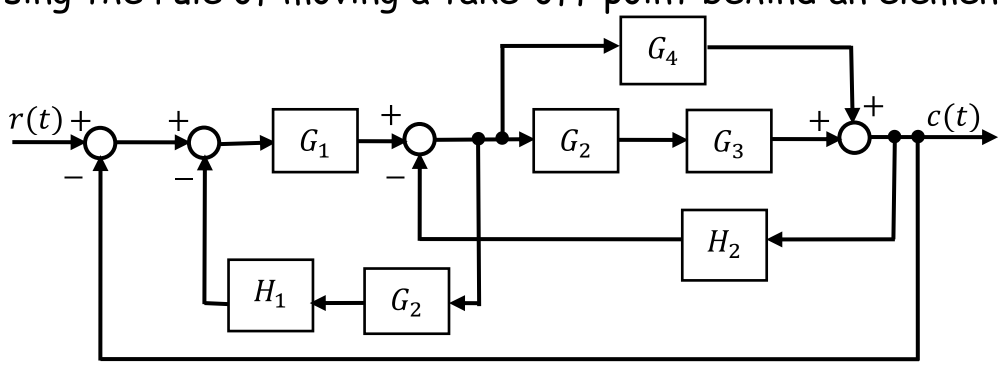
Finally the transfer function of the overall control system is,

$$\frac{c(t)}{r(t)}=\frac{\frac{G_1G_2G_3G_4}{(1+H_1G_1G_2)(1+H_2G_3G_4)}}{1+\frac{H_3G_1G_2G_3G_4}{G_1G_4(1+H_1G_1G_2)(1+H_2G_3G_4)}}=\frac{G_1G_4}{\frac{1}{G_2G_3}(1+H_1G_1G_2)(1+H_2G_3G_4)+H_3}$$

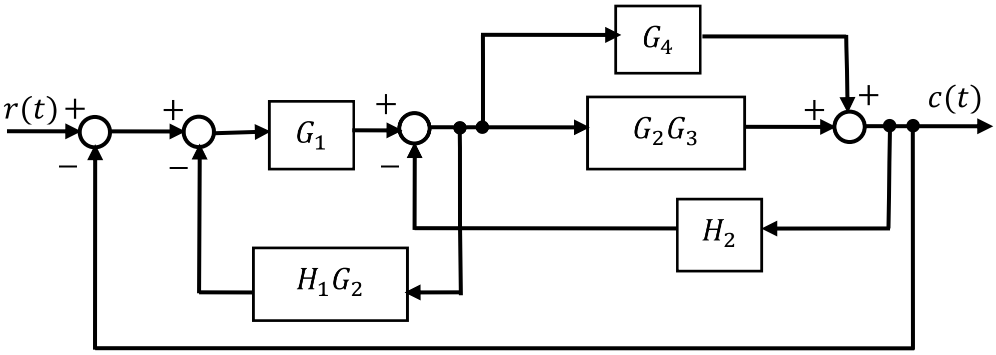
Example 4: Use block diagram reduction techniques and find closed loop transfer function of the system whose block diagram is shown in **Figure**.



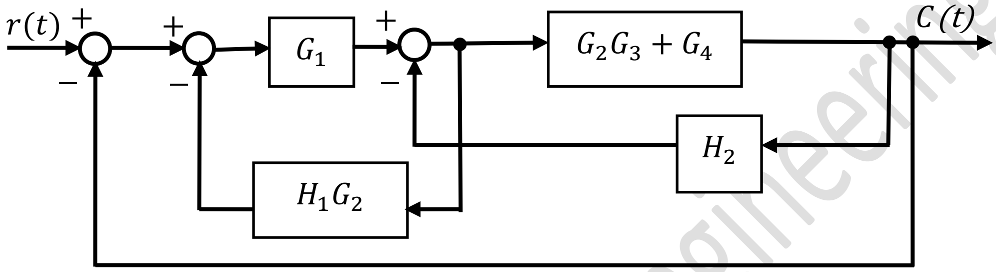
1. Using the rule of moving a take-off point behind an element



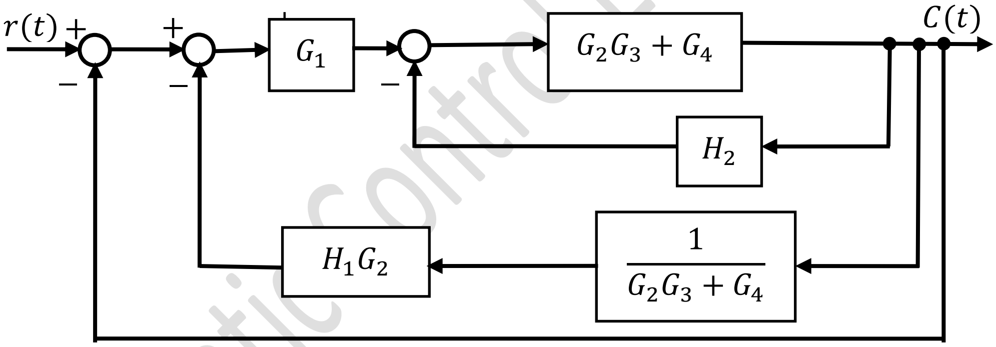
2. Using the rule of combining blocks in cascade



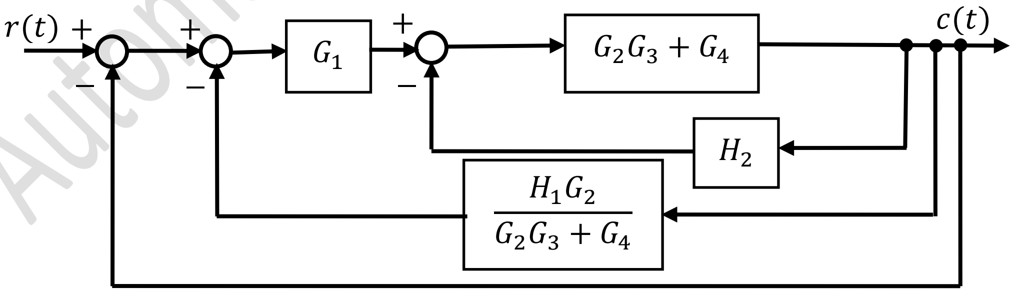
3. Using the rule of combining blocks in parallel



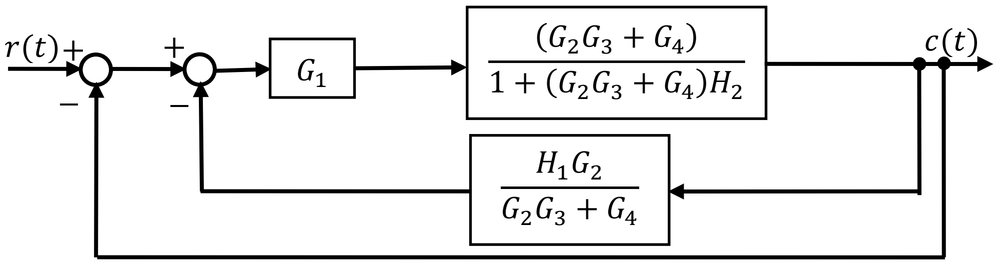
4. Using the rule of moving a take-off point a head of an element



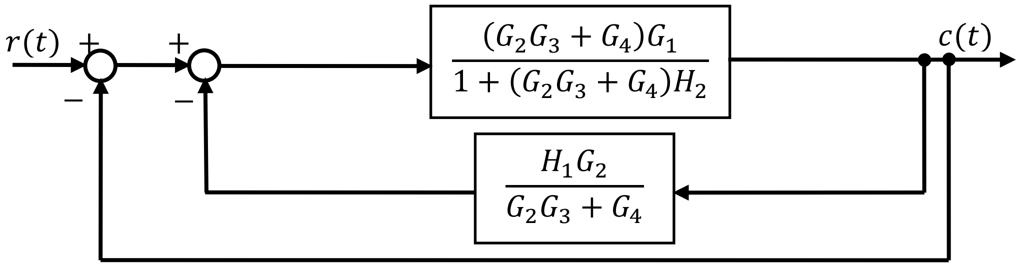
5. Using the rule of combining blocks in cascade



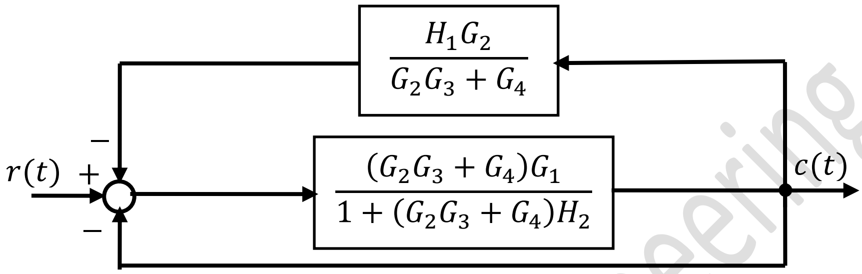
6. Using the rule of eliminating feedback loop



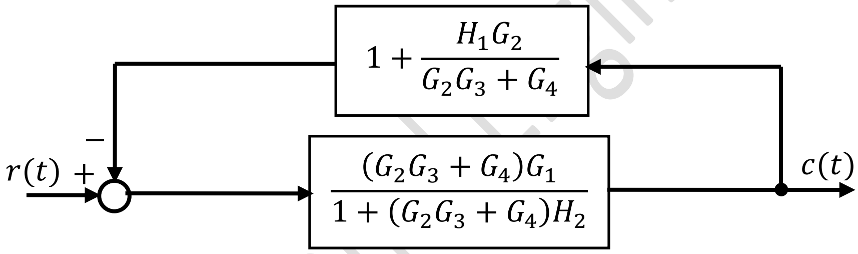
7. Using the rule of combining blocks in cascade



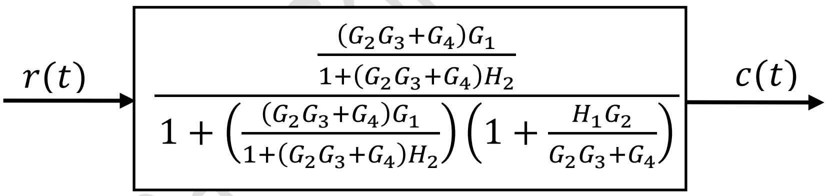
8. Using the rule of combining interconnected summing points



9. Using the rule of combining blocks in parallel



10. Using the rule of eliminating feedback loop

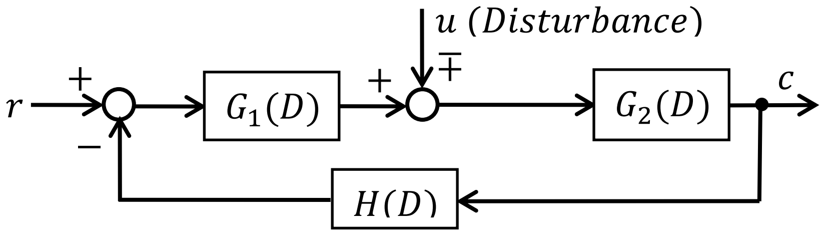


$$\frac{\frac{(G_2G_3+G_4)G_1}{1+(G_2G_3+G_4)H_2}}{1+\left(\frac{(G_2G_3+G_4)G_1}{1+(G_2G_3+G_4)H_2}\right)\left(1+\frac{H_1G_2}{G_2G_3+G_4}\right)}$$

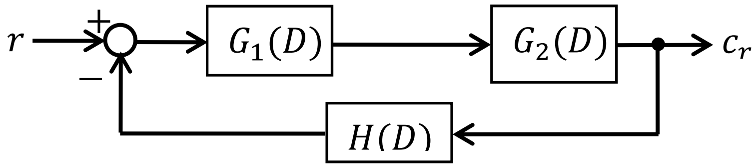
$$\frac{c(t)}{r(t)}=\frac{(G_2G_3+G_4)G_1}{1+(G_2G_3+G_4)(H_2+G_1)+G_1G_2H_1}$$ Transfer Function

Block Diagram Reduction for Multiple Input Single Output (MISO):

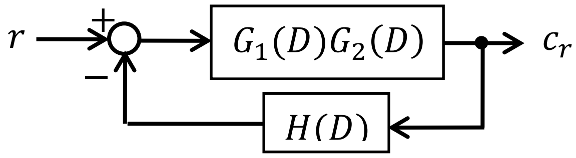
Superposition Method is used to obtain the mathematical differential equation for multiple inputs single output (MISO) control system by setting all inputs except one equal to zero. Consider feedback control system with two inputs and one output as shown in **Figure**,



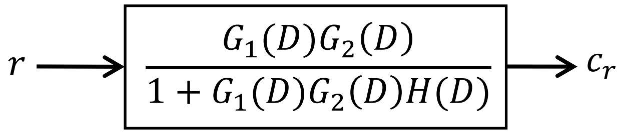
1. Set the disturbance input $u = 0$



2. Using the rule of combining blocks in cascade

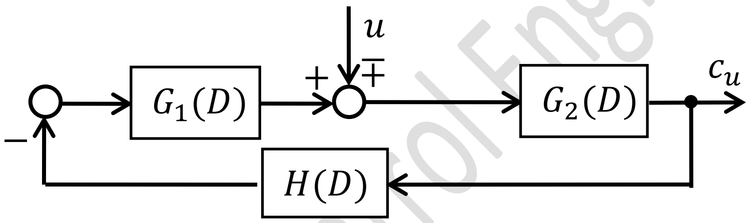


3. Using the rule of eliminating feedback loop

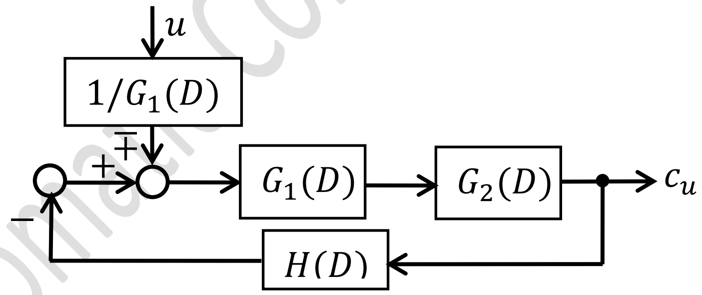


$$c_r = \frac{G_1(D)G_2(D)}{1 + G_1(D)G_2(D)H(D)} r \qquad \frac{c_r}{r} = \frac{G_1(D)G_2(D)}{1 + G_1(D)G_2(D)H(D)}$$

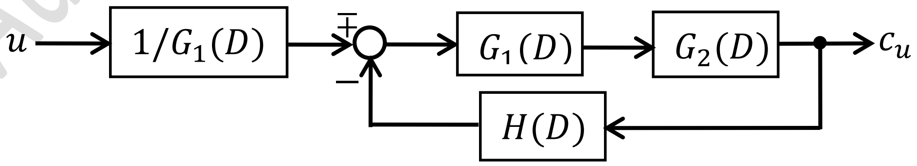
4. Set the reference input $r = 0$



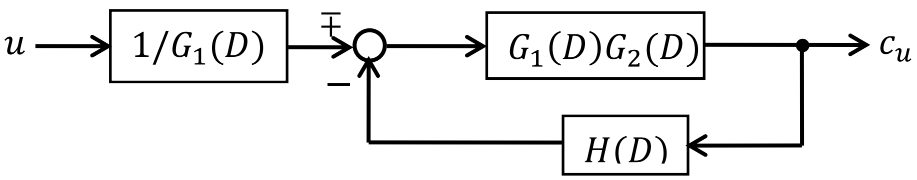
5. Using the rule of moving a summing point behind an element



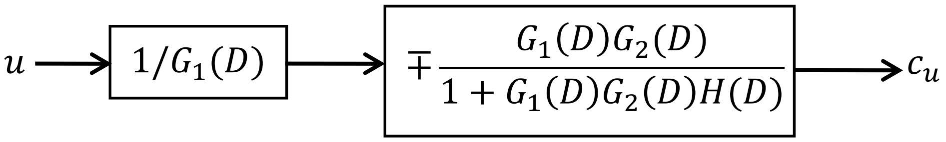
6. Using the rule of combining interconnected summing points



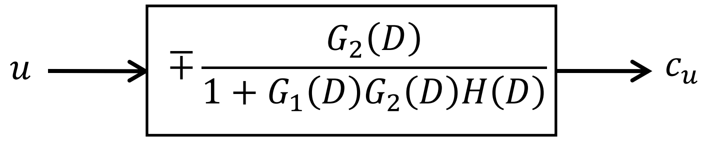
7. Using the rule of combining blocks in cascade



8. Using the rule of eliminating feedback loop



9. Using the rule of combining blocks in cascade,



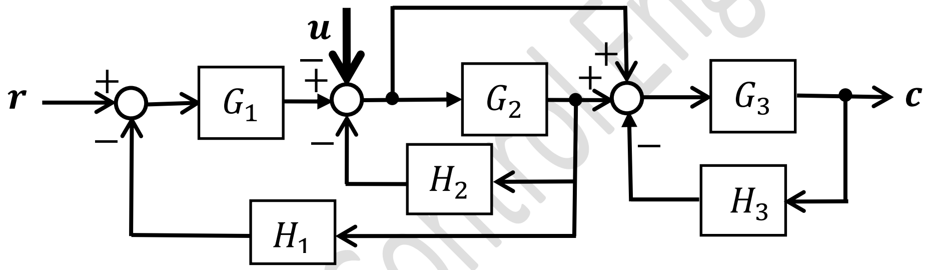
$$c_u = \mp\frac{G_2(D)}{1+G_1(D)G_2(D)H(D)}u \qquad \frac{c_u}{u} = \mp\frac{G_2(D)}{1+G_1(D)G_2(D)H(D)}$$

Principles of superposition method, $c = c_r + c_u$

$$c = \frac{G_1(D)G_2(D)r}{1+G_1(D)G_2(D)H(D)} \mp \frac{G_2(D)u}{1+G_1(D)G_2(D)H(D)} = \frac{G_1(D)G_2(D)r \mp G_2(D)u}{1+G_1(D)G_2(D)H(D)}$$

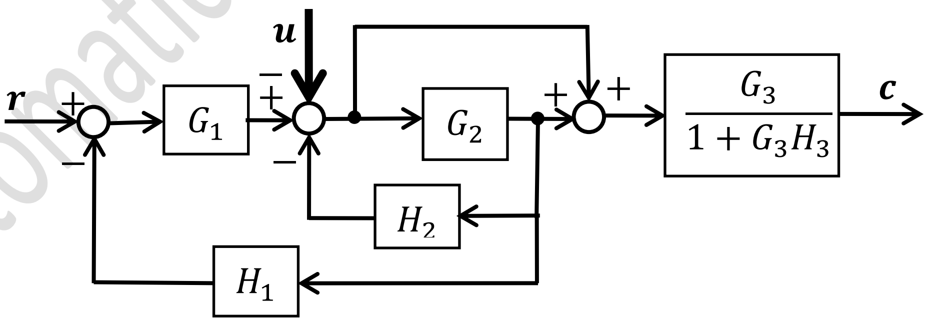
Example 5:

Use techniques of block diagram reduction to find the closed loop transfer function of the control system shown in **Figure**.

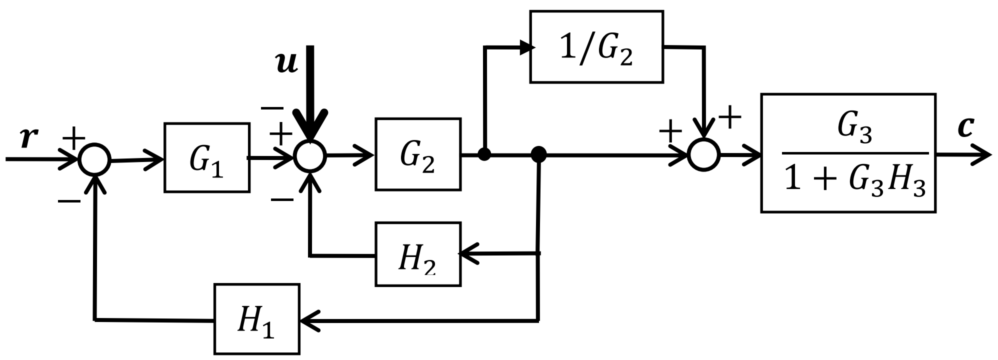


Solution:

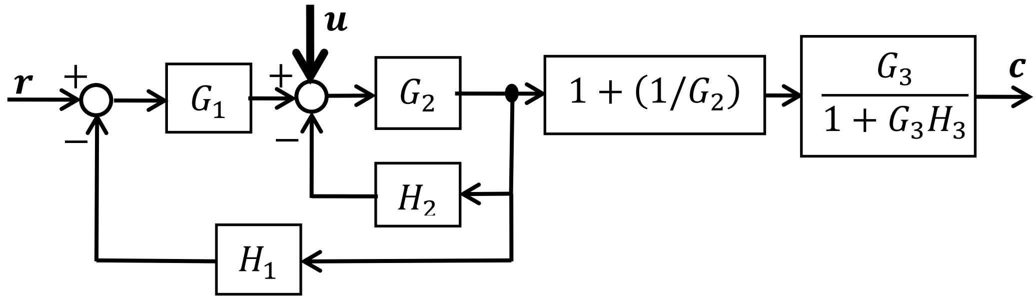
1. Using the rule of eliminating a feedback loop.



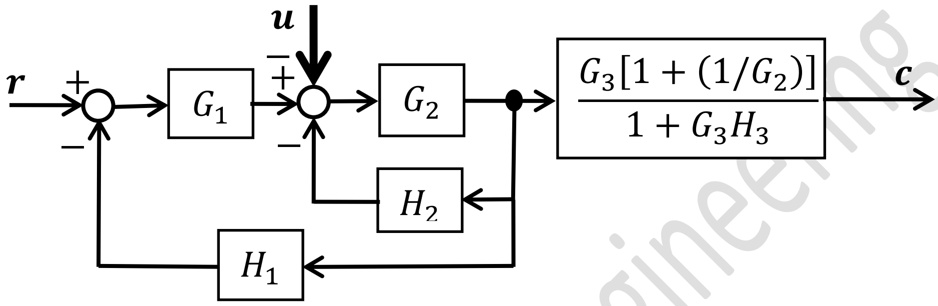
2. Using the rule of moving a take-off point a head of an element



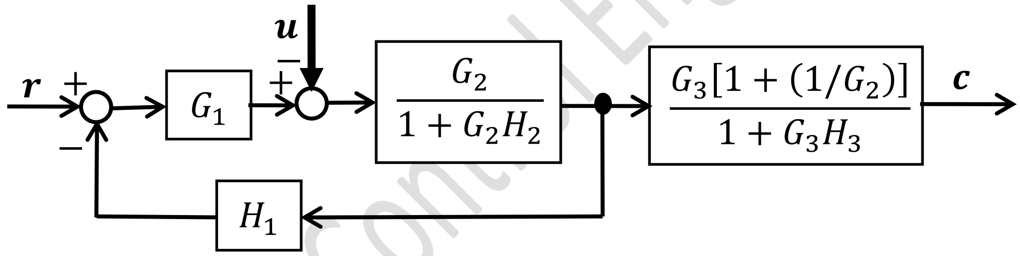
3. Using the rule of combining blocks in parallel



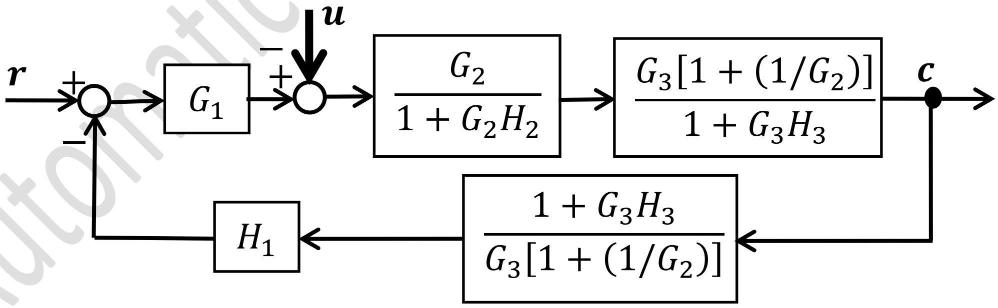
4. Using the rule of combining blocks in cascade



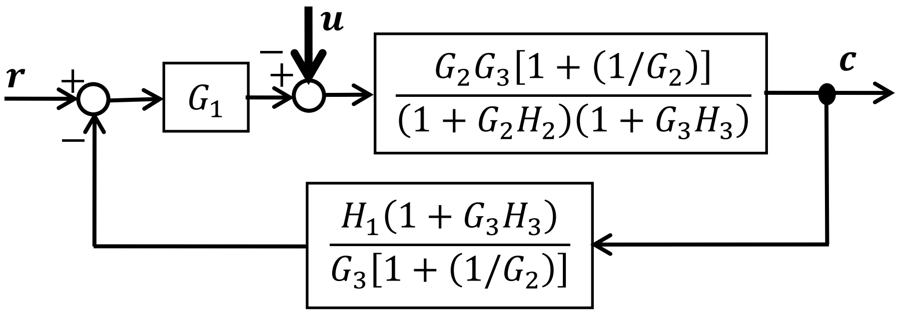
5. Using the rule of eliminating a feedback loop



6. Using the rule of moving a take-off point a head of an element

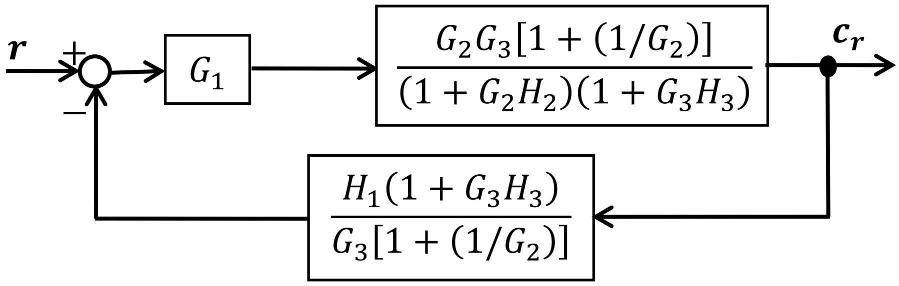


7. Using the rule of combining blocks in cascade

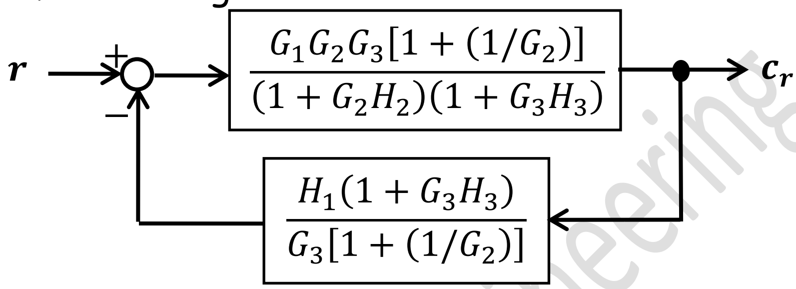


Using method of superposition,

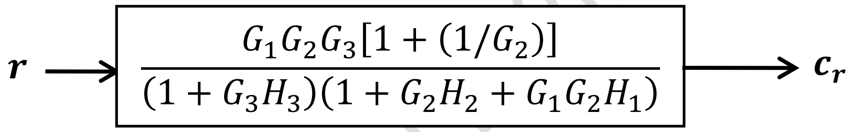
8. Set the disturbance input $u = 0$



9. Using the rule of combining blocks in cascade

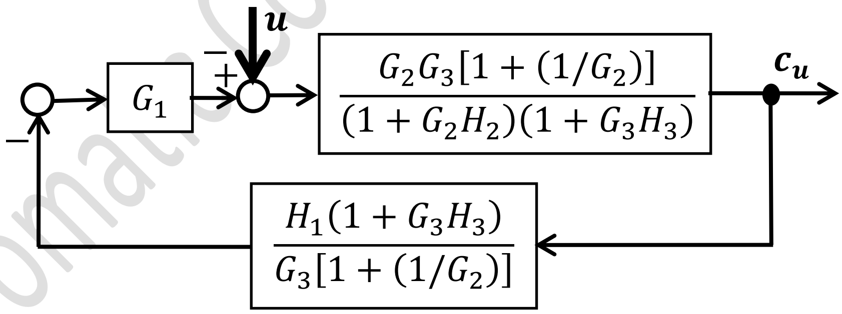


10. Using the rule of eliminating a feedback loop

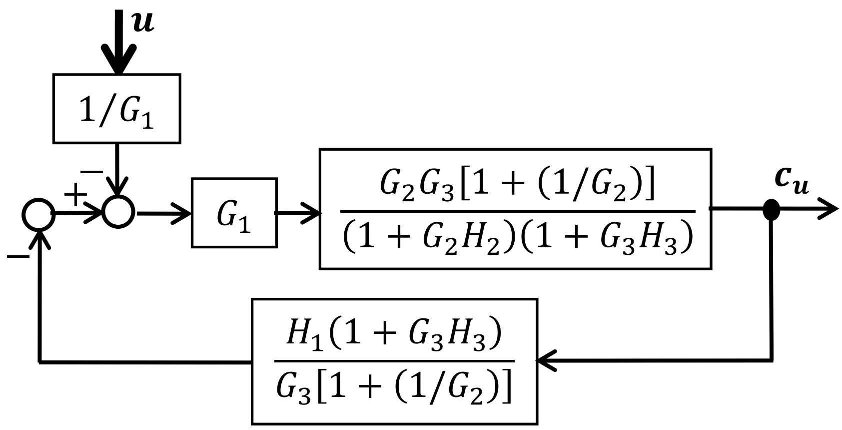


$$c_r = \frac{G_1G_2G_3[1+(1/G_2)]}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)} r$$

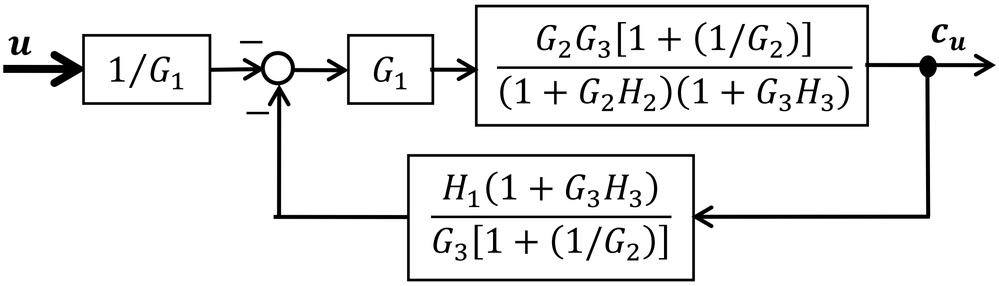
11. Set the reference input $r = 0$



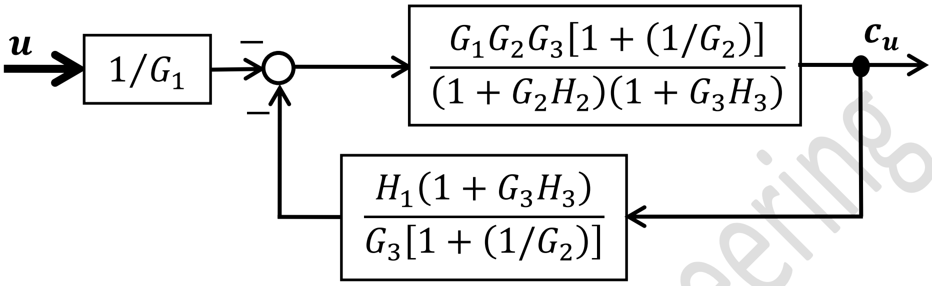
12. Using the rule of moving a summing point behind an element



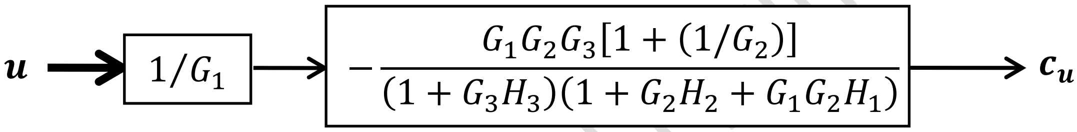
13. Using the rule of combining interconnected summing points



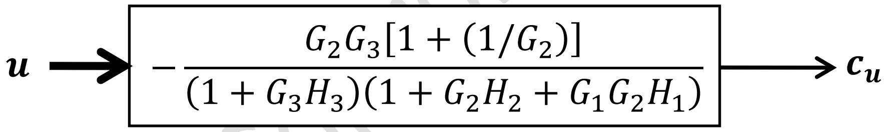
14. Using the rule of combining blocks in cascade



15. Using the rule of eliminating a feedback loop



16. Using the rule of combining blocks in cascade



$$c_u = -\frac{G_2G_3[1+(1/G_2)]}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)}u$$

Principles of superposition method, $c = c_r + c_u$

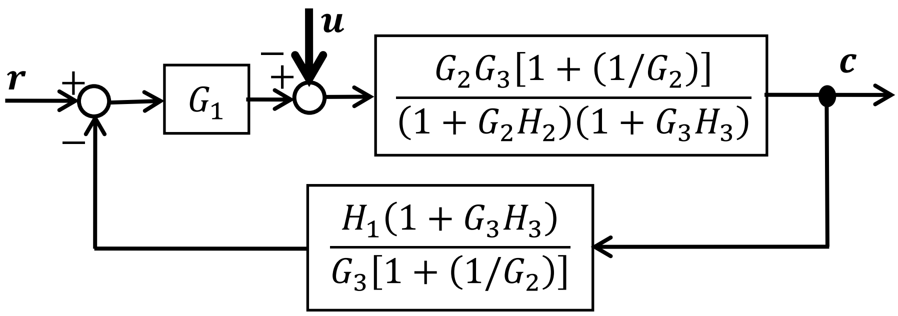
$$c = \frac{G_1G_2G_3[1+(1/G_2)]}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)}r - \frac{G_2G_3[1+(1/G_2)]}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)}u$$

Mathematical differential equation of operation,

$$c = \frac{G_1G_2G_3[1+(1/G_2)]r - G_2G_3[1+(1/G_2)]u}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)}$$

Also using general equation of operation for two inputs and single output,

$$c(t) = \frac{N_{G1}N_{G2}D_H r(t) \mp N_{G2}D_H D_{G1} d(t)}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}$$

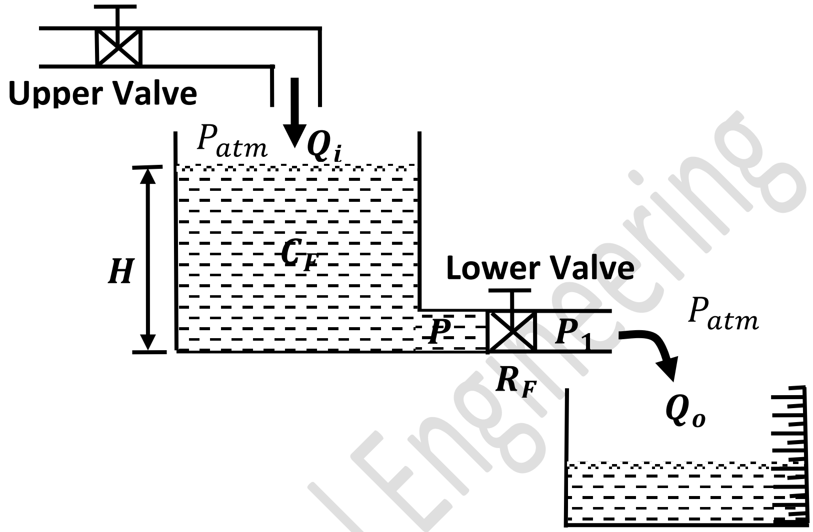


$$c = \frac{G_1G_2G_3[1+(1/G_2)]G_3[1+(1/G_2)]r - G_2G_3[1+(1/G_2)]G_3[1+(1/G_2)]u}{G_1G_2G_3[1+(1/G_2)]H_1(1+G_3H_3)+(1+G_2H_2)(1+G_3H_3)G_3[1+(1/G_2)]}$$

$$C = \frac{G_1G_2G_3[1+(1/G_2)]r - G_2G_3[1+(1/G_2)]u}{(1+G_3H_3)(1+G_2H_2+G_1G_2H_1)}$$

II. Representation of Incompressible Fluid Systems:

Flow of incompressible fluid through a restriction shown in **Figure**,



1. Volume rate of flow at which liquid is stored in upper tank,

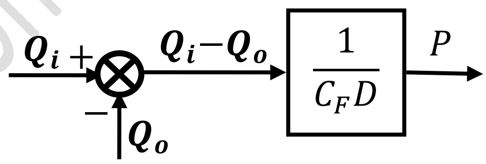
$$Q_i - Q_o = ADH \qquad Q_i - Q_o = \frac{A}{\rho}DP$$

Let $\frac{A}{\rho} = C_F$ Equivalent fluid capacitance

$$Q_i - Q_o = C_F DP$$

Since $(Q_i - Q_o)$ is input while and pressure is output,

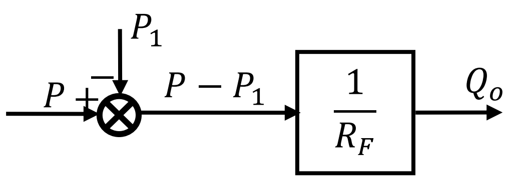
$$P = \frac{1}{C_F D}(Q_i - Q_o)$$



2. Out volume rate of flow Q_o is proportional to pressure difference across lower valve,

$$Q_o = \frac{1}{R_F}(P - P_1)$$

Since $P - P_1$ is input and Q_o is output,

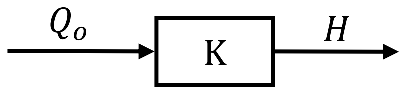


3. Since, $P = \rho H + P_{atm}$ $P_1 = P_{atm}$ $P = \rho H + P_1$ $P - P_1 = \rho H$

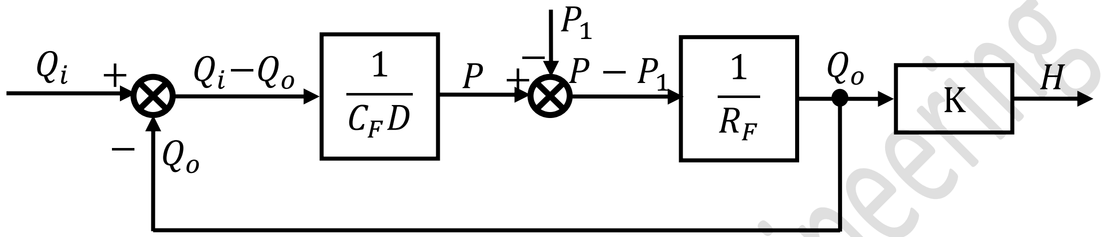
$$Q_o = \frac{1}{R_F}\,(P - P_1) = \frac{1}{R_F}\,\rho H = \frac{\rho}{R_F}\,H$$

Let $K = \frac{R_F}{\rho}$ and $Q_o = \frac{H}{K}$ H is output and Q_o is input,

$H = KQ_o$

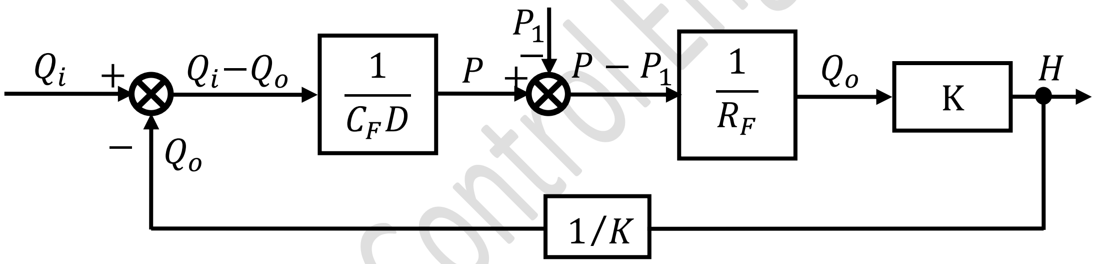


Overall block diagram representation of incompressible fluid flow should be obtained by combining the previous individual block diagrams:

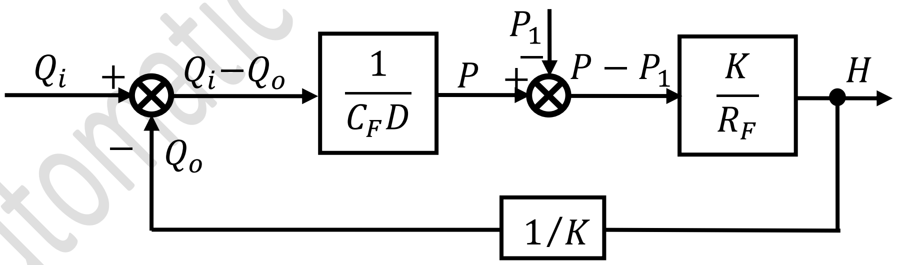


Use techniques of block diagram reduction to determine transfer function,

1. Using the rule of moving a take-off point a head of an element



2. Using the rule of combining blocks in cascade



Mathematical differential equation of operation for flow of incompressible fluid could be obtained using general equation of operation,

$$c(t) = \frac{N_{G1}N_{G2}D_H r(t) \mp N_{G2}D_H D_{G1} d(t)}{N_{G1}N_{G2}N_H \mp D_{G1}D_{G2}D_H}$$

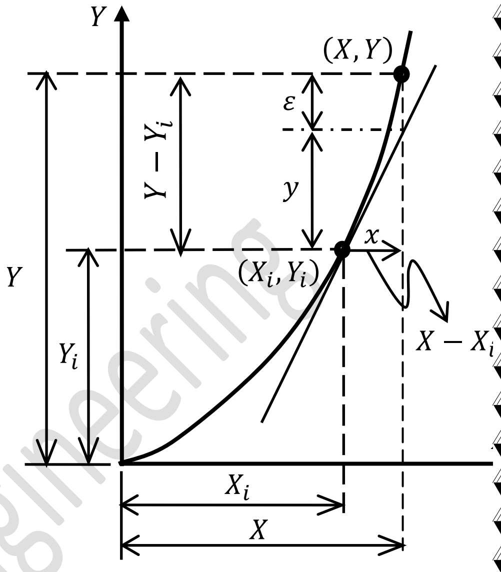
$$H = \frac{KKQ_i - KKC_F D P_1}{K + KR_F C_F D} = \frac{Q_i - C_F D P_1}{\frac{1}{K}(1 + R_F C_F D)}$$

Since, $R_F C_F$ is equal to $\boldsymbol{\tau_s}$ which is the time constant in control fluid system,

$$H = \frac{Q_i - C_F D P_1}{\frac{1}{K}(1 + \tau_s D)} \qquad H = \frac{1}{\frac{1}{K}(1 + \tau_s D)} Q_i - \frac{C_F D}{\frac{1}{K}(1 + \tau_s D)} P_1$$

Linearization of Nonlinear Functions:

Most powerful methods of system analysis are developed for linear control systems. In fact, actual control systems contain some nonlinear elements which in turn yield nonlinear differential equation of operation. Then engineering nonlinear relationships need to be linearized. Consider a nonlinear function as shown in **Figure**,



$$Y = Y(X)$$

$$Y - Y_i = \left.\frac{dY}{dX}\right|_i (X - X_i) \qquad \Delta Y = \left.\frac{dY}{dX}\right|_i \Delta X$$

$$y = \left.\frac{dY}{dX}\right|_i x$$

General procedure for linearization:

$$Y = Y(X_1, X_2, \ldots\ldots\ldots, X_n)$$

$$\Delta Y = \left.\frac{\partial Y}{\partial X_1}\right|_i \Delta X_1 + \left.\frac{\partial Y}{\partial X_2}\right|_i \Delta X_2 + \cdots\cdots + \left.\frac{\partial Y}{\partial X_n}\right|_i \Delta X_n$$

Variations of variables about reference values,

$$Y - Y_i = \left.\frac{\partial Y}{\partial X_1}\right|_i (X_1 - X_{1i}) + \left.\frac{\partial Y}{\partial X_2}\right|_i (X_2 - X_{2i}) + \cdots\cdots + \left.\frac{\partial Y}{\partial X_n}\right|_i (X_n - X_{ni})$$

$$y = \left.\frac{\partial Y}{\partial X_1}\right|_i x_1 + \left.\frac{\partial Y}{\partial X_2}\right|_i x_2 + \cdots\cdots + \left.\frac{\partial Y}{\partial X_n}\right|_i x_n$$

Evaluation of partial derivatives at reference condition yields constants,

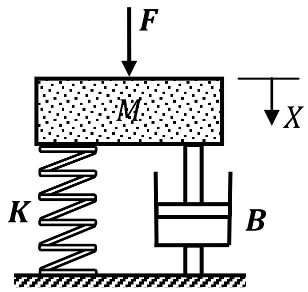
$$c_1 = \left.\frac{\partial Y}{\partial X_1}\right|_i \qquad c_2 = \left.\frac{\partial Y}{\partial X_2}\right|_i \qquad c_n = \left.\frac{\partial Y}{\partial X_n}\right|_i$$

Linear approximation,

$$y = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Example 6. Determine the linearized representation of mass-spring damper differential equation of operation,

$$F = (MD^2 + BD + K)X - Mg$$



Solution:

$$F = F(X)$$

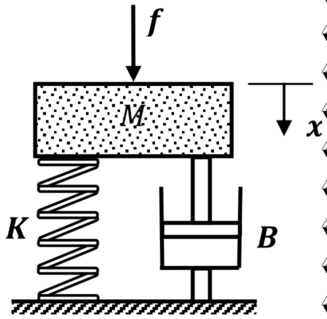
Linearization yields,

$$\Delta F = \left.\frac{\partial F}{\partial X}\right|_i \Delta X$$

Variations of variables from reference values,

$$F - F_i = \left.\frac{\partial F}{\partial X}\right|_i (X - X_i)$$

$$f = \frac{\partial F}{\partial X}\bigg|_i x, \qquad \Rightarrow \qquad f = c_1 x$$



At reference condition,

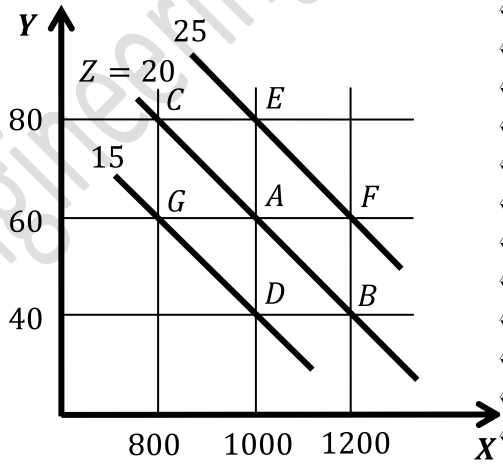
$$c_1 = \frac{\partial F}{\partial X}\bigg|_i = \frac{d}{dX}[(MD^2 + BD + K)X - Mg]\bigg|_i = MD^2 + BD + K$$

$$f = (MD^2 + BD + K)x$$

Linearization of Operating Curves:

Operating characteristics of many components in control systems are given in the form of operating curves rather than equations. A family of operating curves of constant values of Z shown in **Figure**,

$$Y = Y(X, Z)$$



Linearization about a reference point gives,

$$\Delta Y = \frac{\partial Y}{\partial X}\bigg|_Z \Delta X + \frac{\partial Y}{\partial Z}\bigg|_X \Delta Z$$

$$Y - Y_i = \frac{\partial Y}{\partial X}\bigg|_Z (X - X_i) + \frac{\partial Y}{\partial Z}\bigg|_X (Z - Z_i)$$

$$y = \frac{\partial Y}{\partial X}\bigg|_Z x + \frac{\partial Y}{\partial Z}\bigg|_X z$$

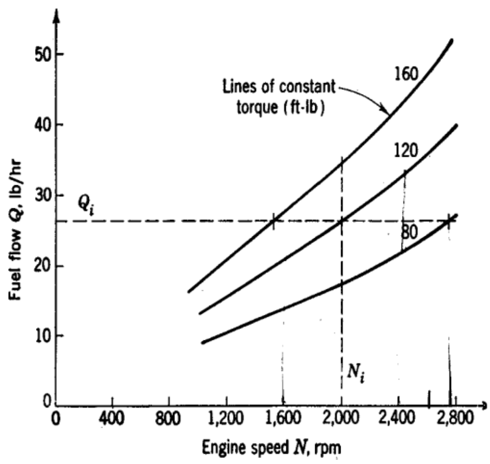
$$\frac{\partial Y}{\partial X}\bigg|_Z = \frac{\Delta Y}{\Delta X}\bigg|_{Z=20} = \frac{Y_B - Y_C}{X_B - X_C} = \frac{40 - 80}{1200 - 800} = -0.1$$

Similarly,

$$\frac{\partial Y}{\partial Z}\bigg|_X = \frac{\Delta Y}{\Delta Z}\bigg|_{X=1000} = \frac{Y_D - Y_E}{Z_D - Z_E} = \frac{40 - 80}{15 - 25} = \frac{-40}{-10} = 4$$

Linearized mathematical differential of operation is, $\quad y = -0.1x + 4z$

Example 6. A typical family of operating curves for an engine is shown in **Figure**. Usually such curves are determined experimentally. Determine the linearized mathematical equation of operation.



Solution:

Speed N is a function of the rate of fuel flow Q and engine torque T,

$$N = N(Q, T)$$

Linearization,

$$n = \frac{\partial N}{\partial Q}\bigg|_T q + \frac{\partial N}{\partial T}\bigg|_Q t$$

Partial derivatives at reference condition,

$$c_1 = \left.\frac{\partial N}{\partial Q}\right|_T \qquad c_2 = \left.\frac{\partial N}{\partial T}\right|_Q \qquad n = c_1 q + c_2 t$$

$$c_1 = \left.\frac{\partial N}{\partial Q}\right|_T = \left.\frac{\Delta N}{\Delta Q}\right|_{T=120} = \frac{2400-1600}{32-20} = 66.7$$

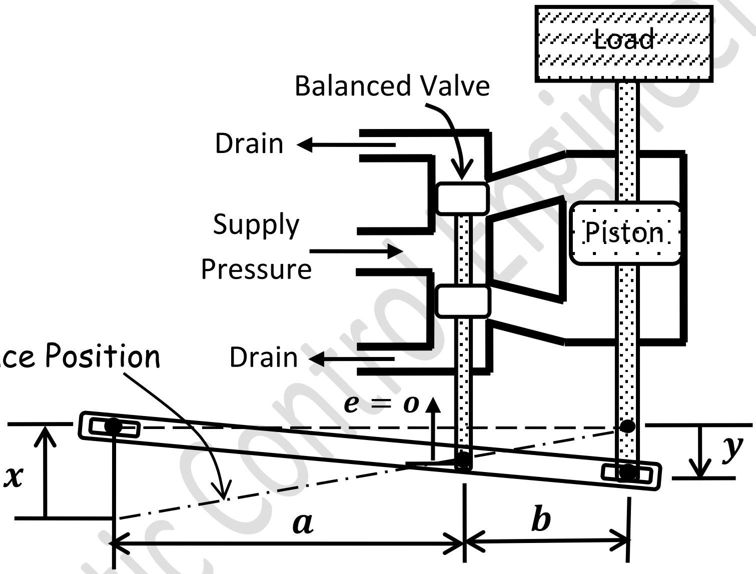
Similarly,

$$c_2 = \left.\frac{\partial N}{\partial T}\right|_Q = \left.\frac{\Delta N}{\Delta T}\right|_{Q=26} = \frac{2730-1530}{80-160} = -15$$

Linearized mathematical equation of operation is, $n = 66.7q - 15t$

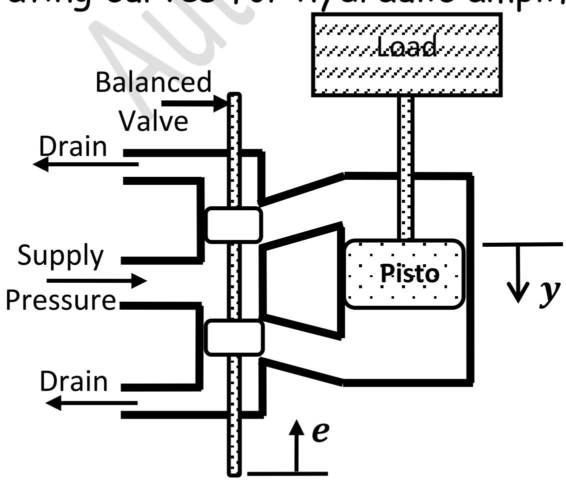
III. Representation of Hydraulic Servomotors:

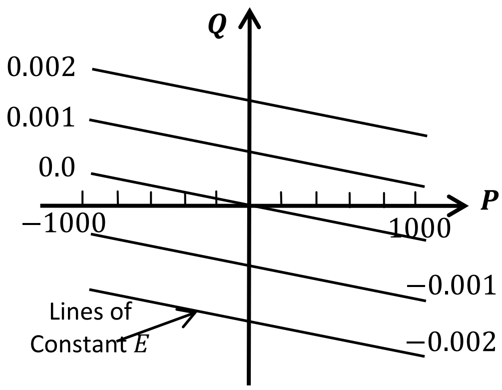
A hydraulic servomotor consists of two components, hydraulic amplifier with walking beam linkage as shown in **Figure**,



1. Hydraulic Amplifier:

Rate of flow Q to the cylinder is a function of valve position E and pressure P drop across the power piston as shown in **Figure** with the operating curves for hydraulic amplifier.





$$Q = Q(E, P)$$

Linearization yields:

$$\Delta Q = \left.\frac{\partial Q}{\partial E}\right|_P \Delta E + \left.\frac{\partial Q}{\partial P}\right|_E \Delta P$$

$$Q - Q_i = \left.\frac{\partial Q}{\partial E}\right|_P (E - E_i) + \left.\frac{\partial Q}{\partial P}\right|_E (P - P_i)$$

$$q = \left.\frac{\partial Q}{\partial E}\right|_P e + \left.\frac{\partial Q}{\partial P}\right|_E p$$

Let, $C_1 = \left.\frac{\partial Q}{\partial E}\right|_P$ and $C_2 = -\left.\frac{\partial Q}{\partial P}\right|_E$ $\qquad q = C_1 e - C_2 p$

Force transmitted to the load by power piston,

$$pA = M\frac{d^2y}{dt^2} = MD^2y \qquad p = \frac{M}{A}D^2y \qquad q = C_1 e - \frac{C_2 M}{A}D^2 y$$

Variation in rate of flow q into a cylinder,

$$q = ADy$$

$$ADy = C_1 e - \frac{C_2 M}{A}D^2 y \qquad \text{let,} \quad \tau = C_2 M/A^2 \qquad y = \frac{(C_1/A)}{D(1+\tau D)}e$$

Position y is output and e is input, transfer function of hydraulic amplifier,

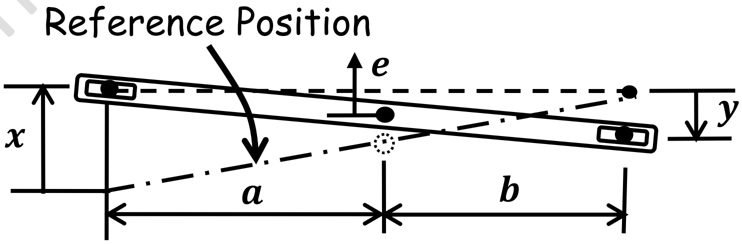
$$\frac{y}{e} = \frac{(C_1/A)}{D(1+\tau D)}$$

Block diagram representation of hydraulic amplifier is:

$$e \rightarrow \boxed{\frac{(C_1/A)}{D(1+\tau D)}} \rightarrow y$$

2. **Walking beam linkage:**

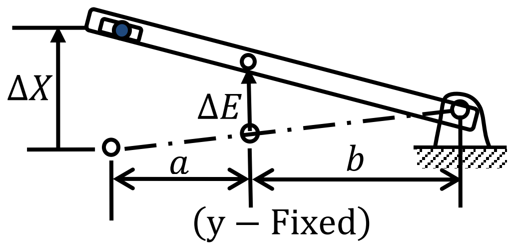
A walking beam linkage connects input (reference) position x, valve position e, and output power piston position y as shown in Figure,



$$E = E(X,Y) \qquad \text{Linearization,} \qquad e = \left.\frac{\partial E}{\partial X}\right|_Y x + \left.\frac{\partial E}{\partial Y}\right|_X y$$

Value of partial derivative $\left.\frac{\partial E}{\partial X}\right|_Y$ is obtained from the **Figure**,

$$\left.\frac{\partial E}{\partial X}\right|_Y = \left.\frac{\Delta E}{\Delta X}\right|_Y = \frac{b}{a+b}$$

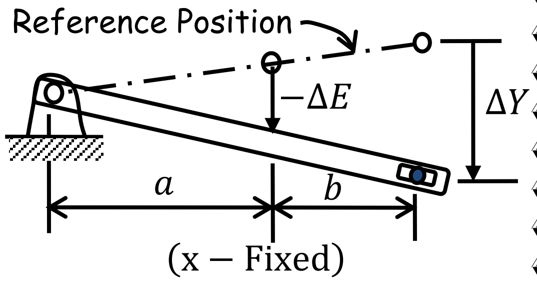


Value of partial derivative $\left.\frac{\partial E}{\partial Y}\right|_X$ is obtained from the **Figure**,

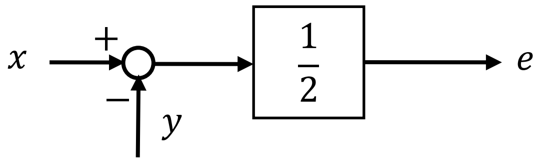
$$\left.\frac{\partial E}{\partial Y}\right|_X = \left.\frac{\Delta E}{\Delta Y}\right|_X = -\frac{a}{a+b}$$

$$e = \frac{b}{a+b}x - \frac{a}{a+b}y$$

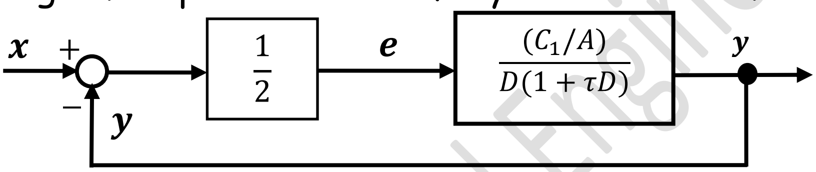
For, $a = b$ $e = \frac{1}{2}(x - y)$



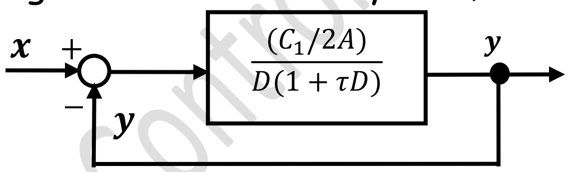
Block diagram representation of walking-beam linkage is:



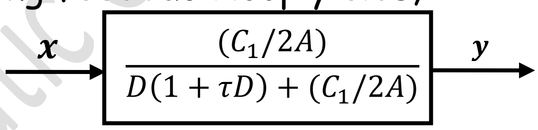
Overall Block Diagram Representation of Hydraulic Servomotors,



Using the rule of combining blocks in cascade yields,



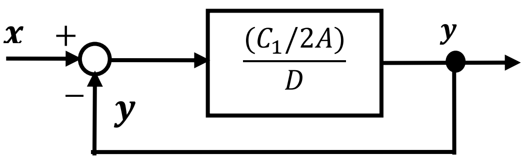
Using the rule of eliminating feedback loop yields,



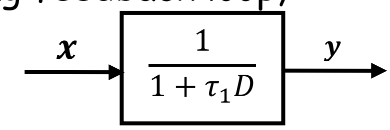
Let $\tau_1 = 2A/C_1$ then closed loop transfer function is:

$$\frac{y}{x} = \frac{1}{1 + \tau_1 D(1+\tau D)}$$

<u>**Note:**</u> For neglected load ($M = 0$), accordingly $\tau = 0$ then,



Using the rule of eliminating feedback loop,

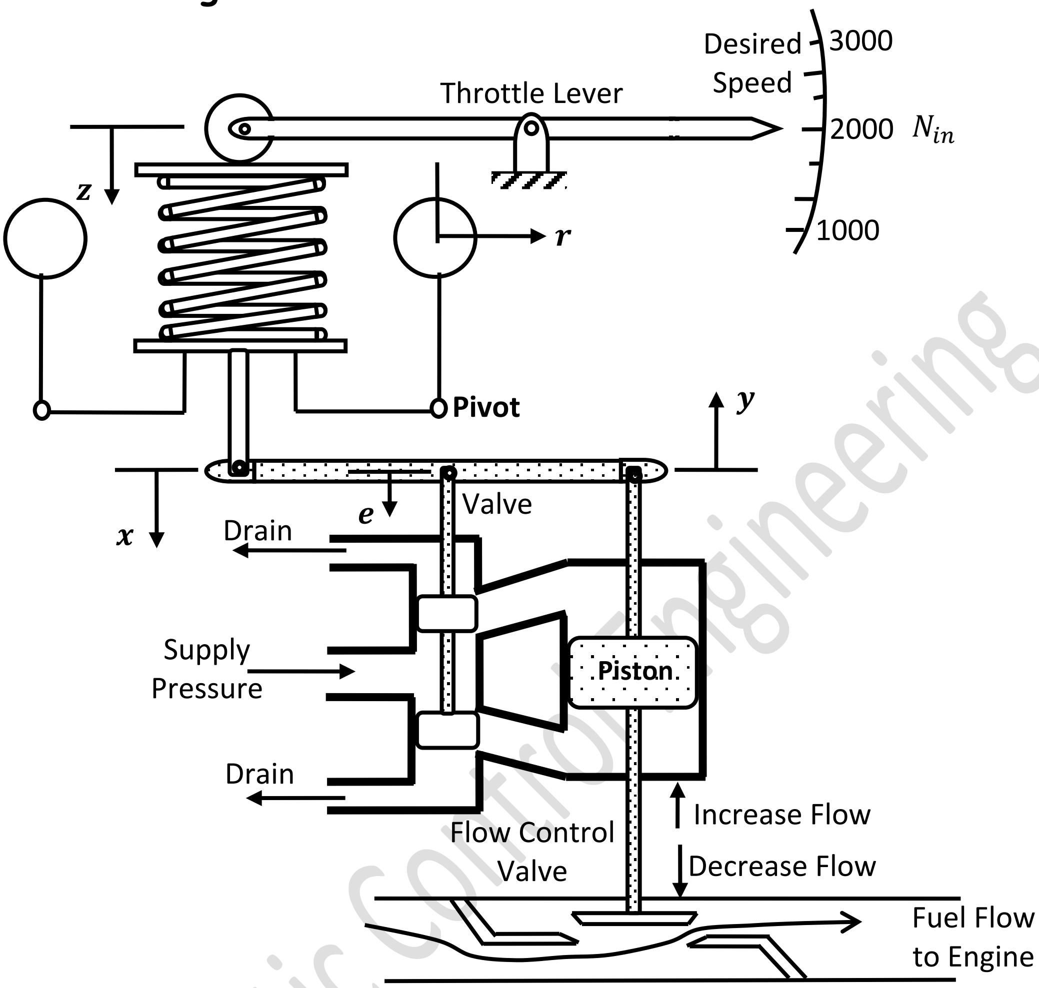


And closed loop transfer function,

$$\frac{y}{x} = \frac{(C_1/2A)}{D + (C_1/2A)} = \frac{1}{1 + (2A/C_1)D} = \frac{1}{1+\tau_1 D}$$

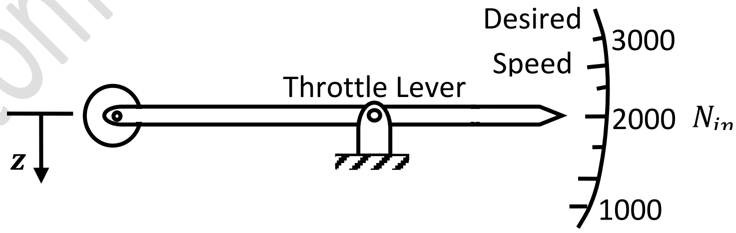
IV. Speed Control Systems:

A controller of speed governor automatically contr ols the speed of an engine as shown in **Figure**.



1. Throttle lever

Position of throttle lever sets desired speed as shown in **Figure**,



Position Z is a function of desired speed:

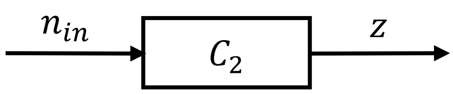
$$Z = Z(N_{in})$$

Linearization,

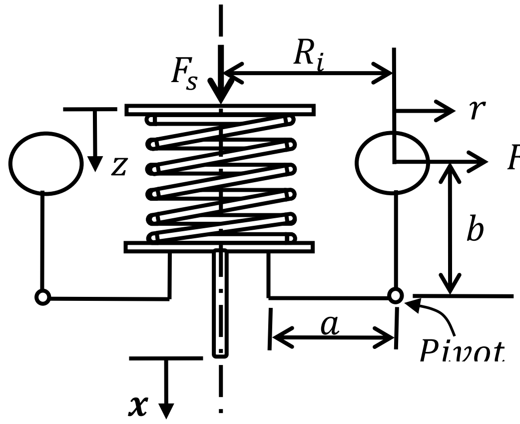
$$z = \left.\frac{\partial Z}{\partial N_{in}}\right|_i n_{in} \qquad \text{let,} \qquad C_2 = \left.\frac{\partial Z}{\partial N_{in}}\right|_i \qquad z = C_2 n_{in}$$

Desired speed n_{in} is input and z is output,

$$\frac{z}{n_{in}} = C_2$$



2. Fly-weight Speed Governor



In fly-weight speed governor shown in **Figure**, centrifugal force is,

$$F_C = MR\omega^2 \qquad \omega = C_g N_O$$

Where, C_g is gear ratio and N_O is output speed,

$$F_C = C_g^2 MRN_O^2$$

Take moments about the pivot,

$$\frac{F_S}{2}a = F_C b \qquad F_S = 2\frac{b}{a}C_g^2 MRN_O^2 \qquad C_f = 2C_g^2 M \qquad C_r = \frac{b}{a}$$

$$F_S = C_f C_r RN_O^2 \qquad F_s = F_s(R, N_O)$$

Linearization,

$$f_S = \left.\frac{\partial F_S}{\partial R}\right|_{N_O} r + \left.\frac{\partial F_S}{\partial N_O}\right|_{R} n_O$$

Let, $C_3 = \left.\frac{\partial F_S}{\partial R}\right|_i = C_f C_r N_{Oi}^2$ and, $C_4 = \left.\frac{\partial F_S}{\partial N_O}\right|_i = 2C_f C_r R_i N_{oi}$

$$f_S = C_3 r + C_4 n_O$$

Variation in force exerted by the spring is:

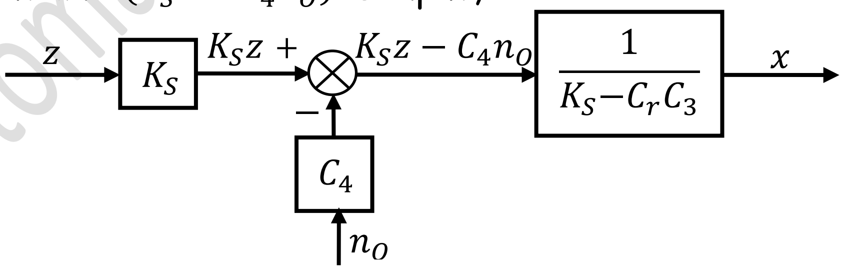
$$f_S = K_S(z - x) \qquad K_S(z - x) = C_3 r + C_4 n_O$$

Since, $r = -\frac{b}{a}x = -C_r x$ $\qquad K_S(z - x) = -C_r C_3 x + C_4 n_O$

Mathematical equation of fly-weight speed governor component,

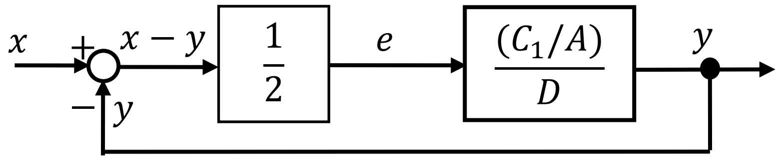
$$x = \frac{1}{K_S - C_r C_3}(K_S z - C_4 n_O)$$

Since x is output and $(K_S z - C_4 n_O)$ is input,



3. Hydraulic Servomotors:

Overall Block Diagram Representation of Hydraulic Servomotors when load is neglected,



4. Control of Fuel Flow:

Flow of fuel through flow valve to the engine is a function of position y,

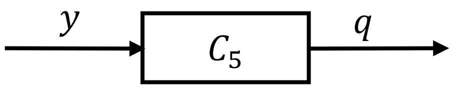
$$Q = Q(Y)$$

Linearized yields,

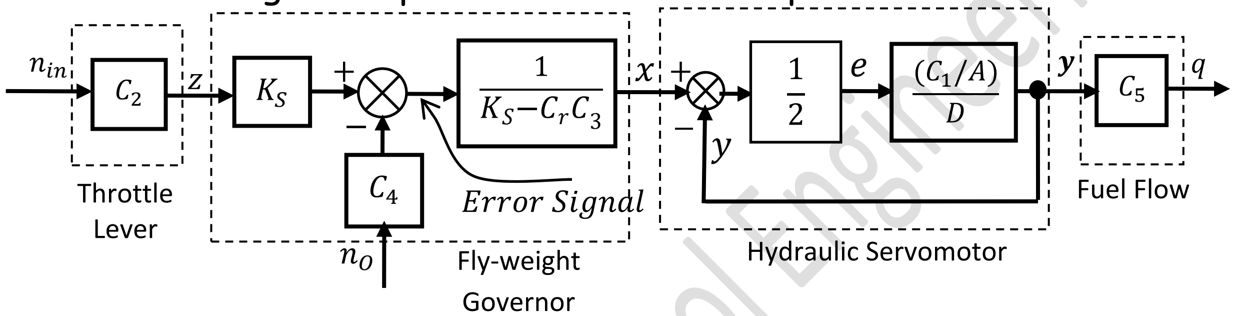
$$q = \left.\frac{\partial Q}{\partial Y}\right|_i y$$

Let, $C_5 = \left.\frac{\partial Q}{\partial Y}\right|_i$ Then, $q = C_5 y$

Position y is input and q is output,



Overall block diagram representation for the speed controller:



In the speed control system for gas turbine of a jet airplane as,

$$N_o = N_o(Q, T)$$

Linearized,

$$n_o = \left.\frac{\partial N_o}{\partial Q}\right|_i q + \left.\frac{\partial N_o}{\partial T}\right|_i t$$

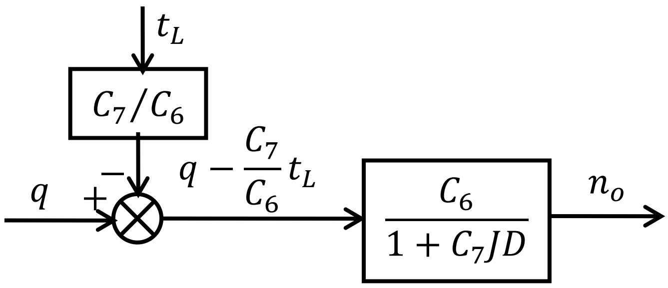
Let $C_6 = \left.\frac{\partial N_o}{\partial Q}\right|_i$ and $C_7 = -\left.\frac{\partial N_o}{\partial T}\right|_i$, then: $n_o = C_6 q - C_7 t$

Torque balance available to accelerate the engine is:

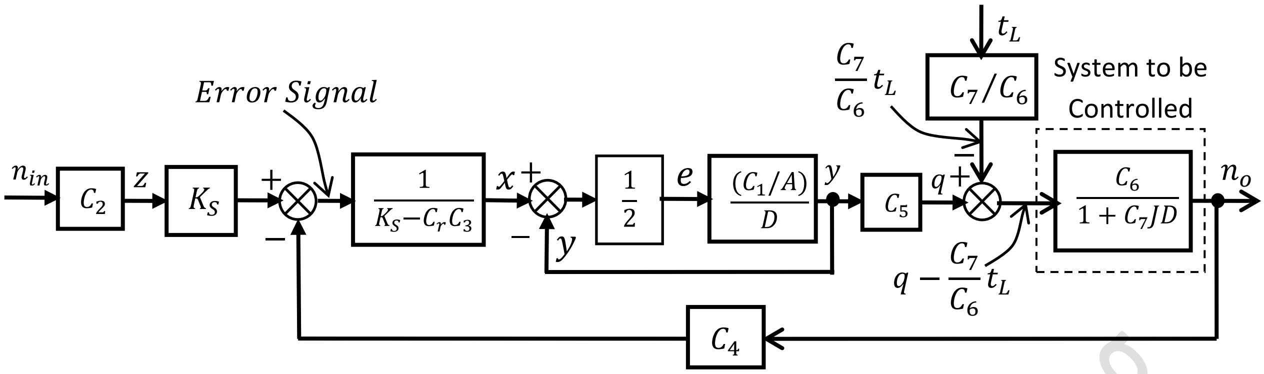
$t - t_L = J\alpha = JDn_o$ $t = JDn_o + t_L$ $n_o = C_6 q - C_7(JDn_o + t_L)$

$$n_o = \frac{C_6 q - C_7 t_L}{1 + C_7 JD} = \frac{C_6}{1 + C_7 JD}\left(q - \frac{C_7}{C_6} t_L\right)$$

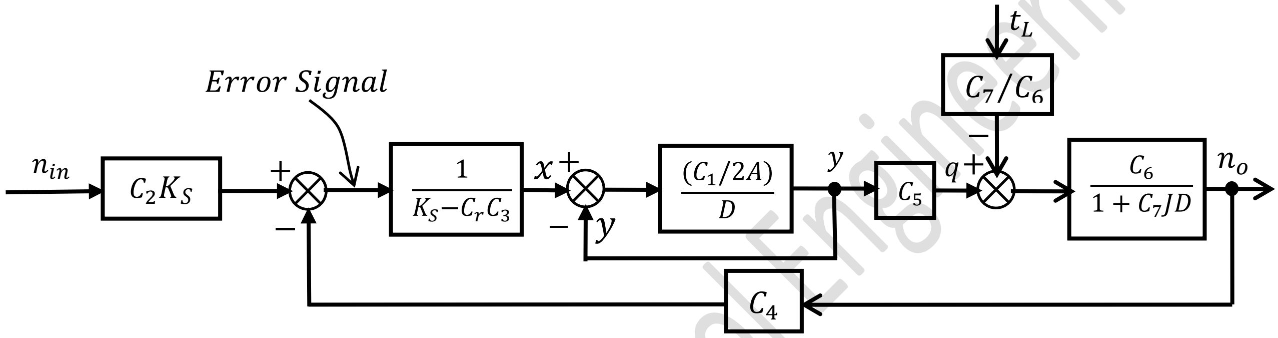
Since n_o is output and $(q - C_8 t_L)$ is input, the block diagram representation shown in **Figure**,



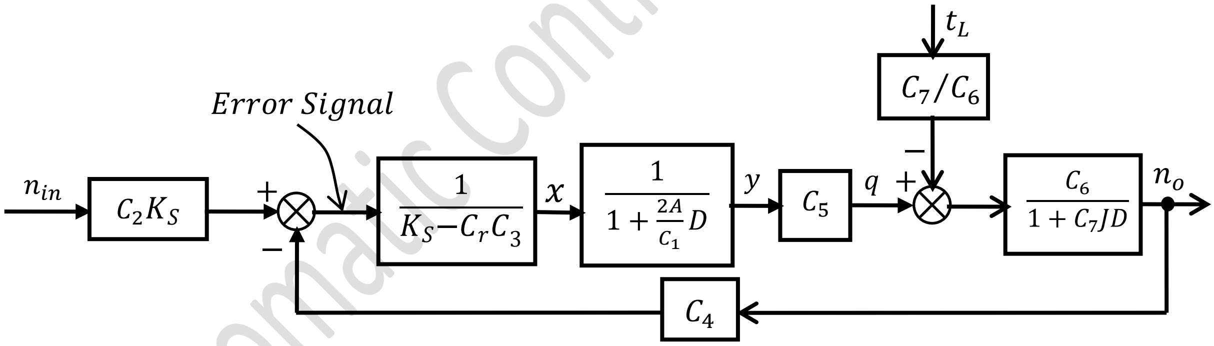
Overall block diagram representation of speed control system for the gas turbine of jet engine,



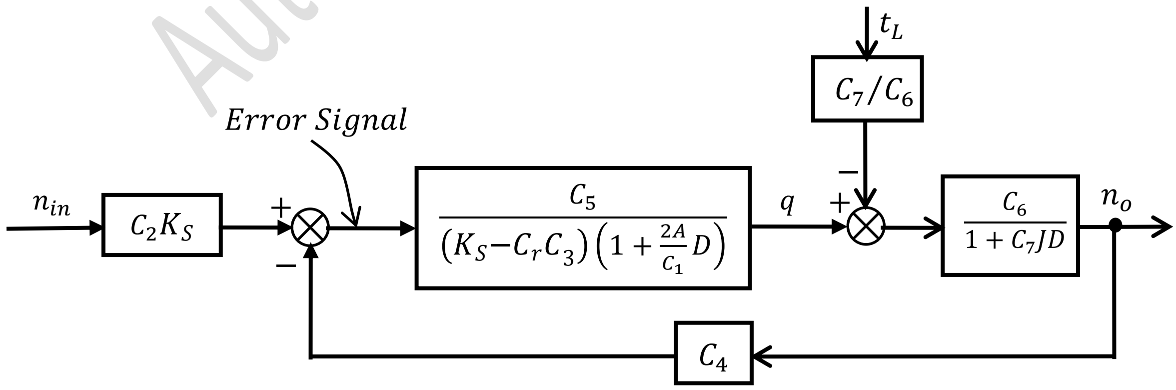
Using the rule of combining blocks in cascade,



Using the rule of eliminating feedback loop,



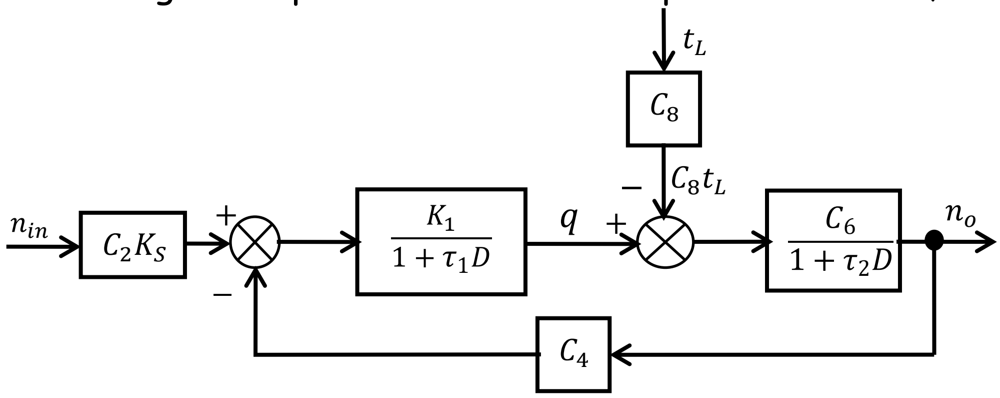
Using the rule of combining blocks in cascade,



Let,

$$K_1 = \frac{C_5}{K_S - C_r C_3}, \qquad \tau_1 = 2A/C_1, \qquad \tau_2 = C_7 J, \qquad C_8 = C_7/C_6$$

Overall block diagram representation of the speed controller,



Using general equation of feedback control system with two inputs and one output,

$$C(t) = \frac{N_{G1}N_{G2}D_H r(t) + N_{G2}D_{G1}D_H d(t)}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}$$

Mathematical differential equation of gas turbine in a jet airplane,

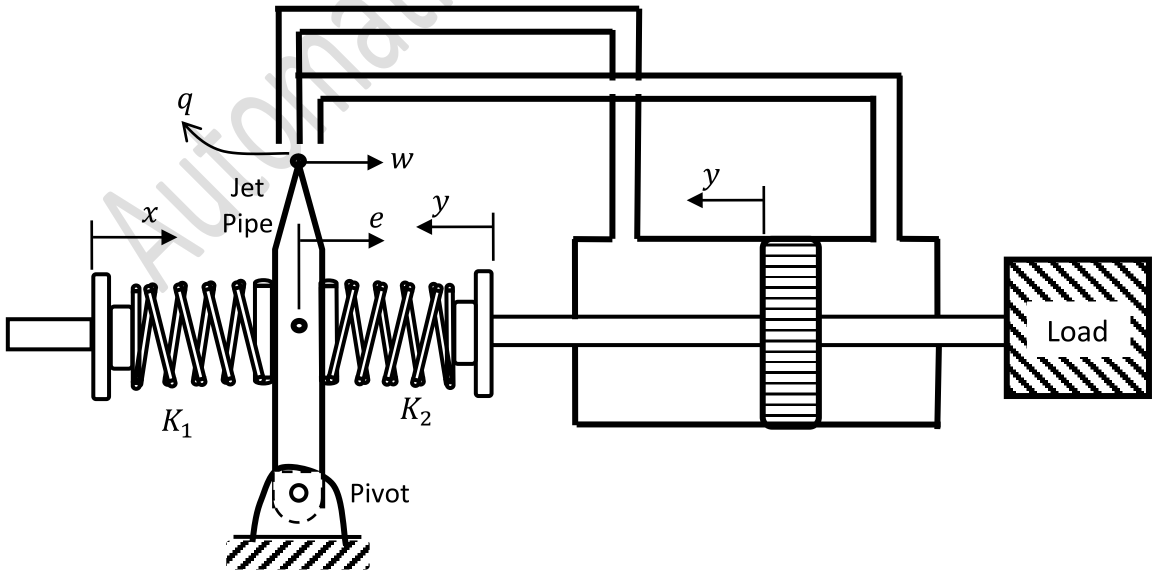
$$n_o = \frac{C_2C_6K_1K_s n_{in} - C_6C_8(1+\tau_1 D)t_L}{C_4C_6K_1 + (1+\tau_1 D)(1+\tau_2 D)}$$

Or,

$$n_o = \frac{K_1C_6C_2K_s}{C_4C_6K_1 + (1+\tau_1 D)(1+\tau_2 D)} n_{in} - \frac{C_6C_8(1+\tau_1 D)}{C_4C_6K_1 + (1+\tau_1 D)(1+\tau_2 D)} t_L$$

V. Representation of Jet Pipe Amplifier:

A schematic diagram of a jet pipe amplifier is shown in **Figure**. It can be easily noticed that the position x is input and y which is the position of the power piston, is output position.

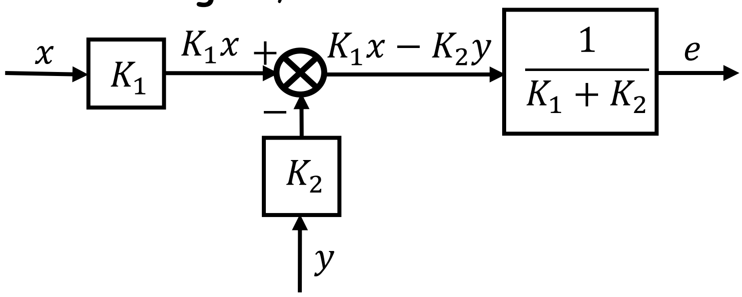


1. Since the compression of spring K_1 is $(x-e)$ and the compression of K_2 is $(e+y)$, then at the reference position,

$$K_1(x-e) = K_2(e+y) \qquad K_1x - K_2y = (K_1+K_2)e$$

$$e = \frac{1}{K_1 + K_2}(K_1 x - K_2 y)$$

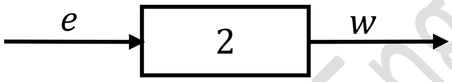
Since e is output and $(K_1 x - K_2 y)$ is input, the block diagram representation shown in **Figure**,



2. For a typical jet pipe, position of jet pipe at the nozzle end w is twice position of jet pipe at centerline of springs e, then:

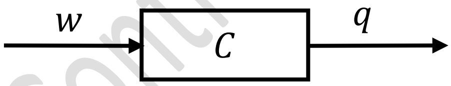
$$w = 2e$$

Since position e is input and w is output, the associated block diagram is,



3. Rate of flow q to the power piston, $Q = Q(w)$, linearization yields,

$q = \left.\frac{\partial Q}{\partial w}\right|_i w$, Let, $\left.\frac{\partial Q}{\partial w}\right|_i = C$ $\qquad q = Cw$

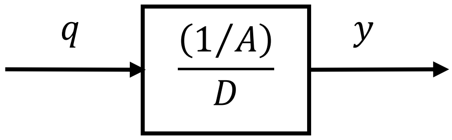


4. Volume rate of flow to the power piston is,

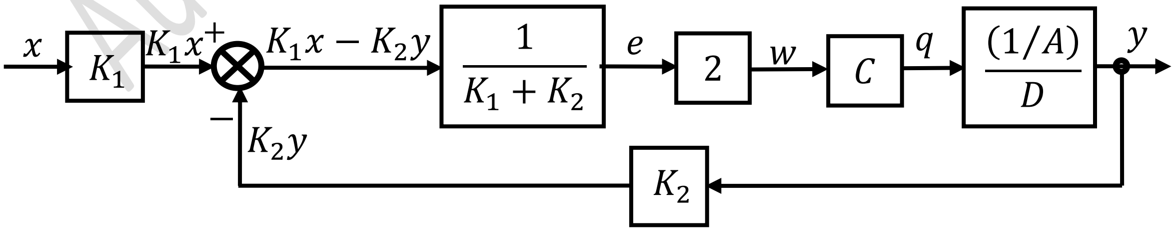
$$q = ADy$$

Since q is input and y is output,

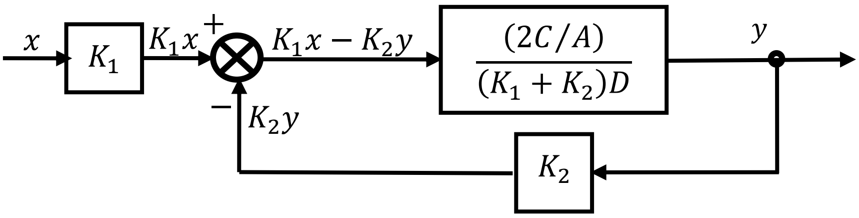
$$y = \frac{(1/A)}{D} q$$



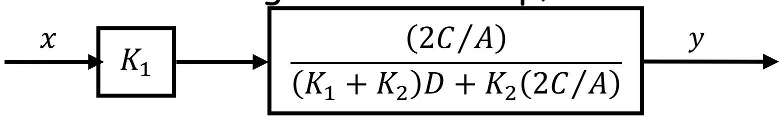
Overall block diagram representation of jet pipe amplifier,



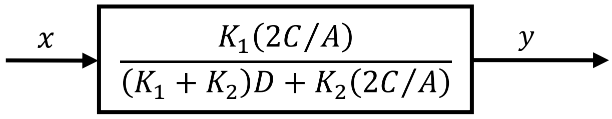
Using the rule of combining blocks in cascade,



Using the rule of eliminating feedback loop,



Using the rule of combining blocks in cascade,



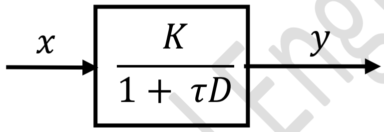
The transfer function is,

$$\frac{y}{x} = \frac{K_1(2C/A)}{(K_1 + K_2)D + K_2(2C/A)} = \frac{K_1/K_2}{1 + (A/2C)[(K_1 + K_2)/K_2]D}$$

Let, $K = K_1/K_2$, and $\tau = (A/2C)[(K_1 + K_2)/K_2]$,

$$\frac{y}{x} = \frac{K}{1 + \tau D}$$

And the block diagram representation for jet pipe amplifier is,



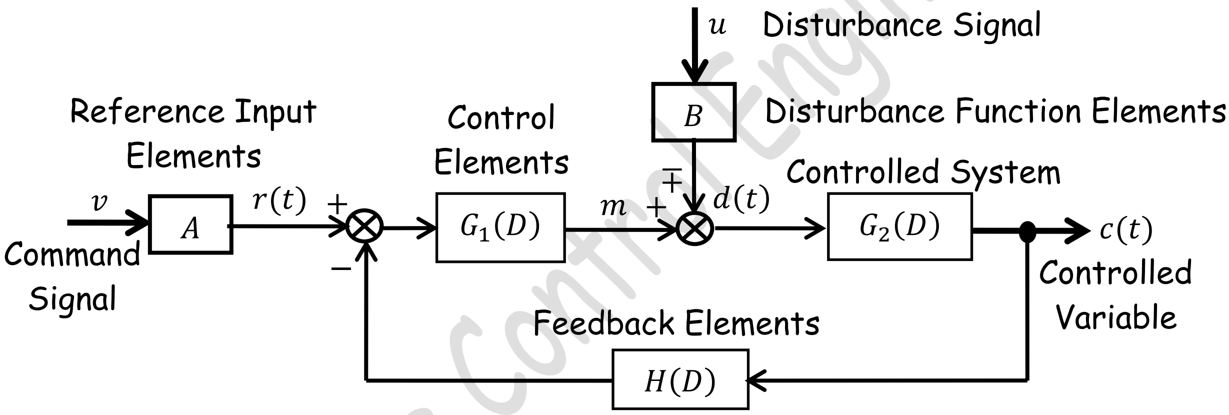
Chapter Four

Steady-State Operation

Operating characteristics of control systems are determined by solving their mathematical differential equations of operation. Transient and steady state operations are two conditions where time response of a control system could be investigated separately. In steady state operation the equilibrium state is attained such that there is no change with respect to time of any system variables. A control system remains at this equilibrium state until it is excited by a change in the desired input or in an external disturbance. A transient condition exists as long as some variable changes with time.

Steady State Equation (Algebraically):

General Block Diagram Representation of a feedback control system with two inputs and one output shown in **Figure**,



General mathematical differential equation of operation,

$$c(t) = \frac{G_1(D)G_2(D)r(t) \mp G_2(D)d(t)}{1 + G_1(D)G_2(D)H(D)}$$

Also it could be expressed as,

$$c(t) = \frac{G_1(D)G_2(D)Av \mp G_2(D)Bu}{1 + G_1(D)G_2(D)H(D)}$$

$$c(t) = \frac{AG_1(D)G_2(D)}{1 + G_1(D)G_2(D)H(D)}v \mp \frac{BG_2(D)}{1 + G_1(D)G_2(D)H(D)}u$$

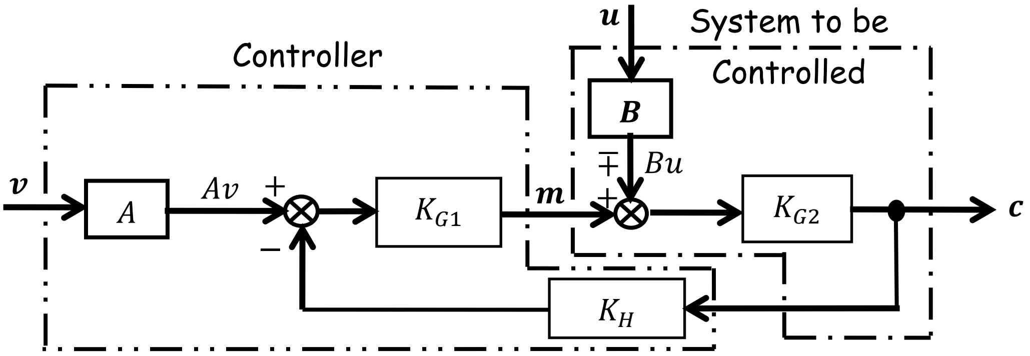
For steady-state operation $(D = 0)$ the steady state constants are,

$$K_{G1} = [G_1(D)]_{D=0} \ , \qquad K_{G2} = [G_2(D)]_{D=0} \ , \qquad K_H = [H(D)]_{D=0}$$

Steady-state equation,

$$c = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H}v + \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H}u$$

General block diagram representation of steady state operation is described as shown **in Figure**,



For $c = v$, coefficient of desired input v must be unity,

$$\frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} = \frac{A}{\frac{1}{K_{G1}K_{G2}} + K_H} = 1$$

Where, constant A is the scale factor of input dial,

$$A = \frac{1}{K_{G1}K_{G2}} + K_H$$

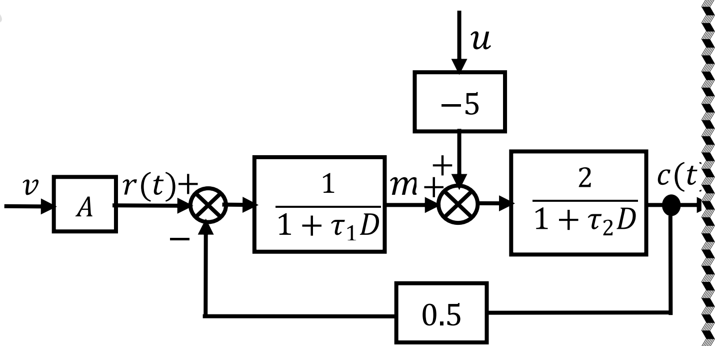
Also, coefficient of external disturbance u is equal to zero,

$$\frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} = \frac{B}{\frac{1}{K_{G2}} + K_{G1}K_H} = 0$$

It is satisfied only when K_{G1} is infinite by an integrator in control elements to give a $(1/D)$ term. This type of controller is called an integral control system.

Example 1:

Determine the steady-state constants and the steady state response for the control system shown in **Figure**. Select constant A such that the coefficient of the desired input v is unity.

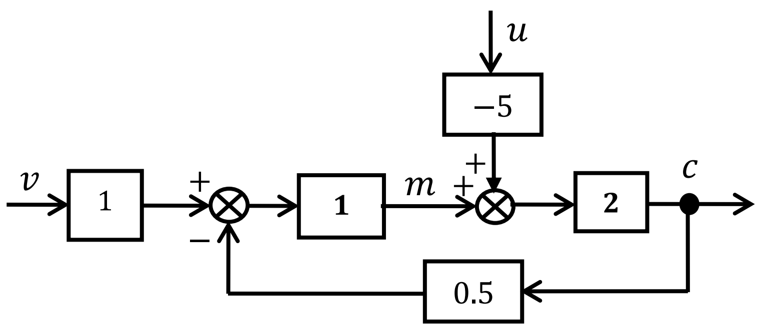


Solution: Steady-state constants are,

$$K_{G1} = \left.\frac{1}{1 + \tau_1 D}\right|_{D=0} = 1, \qquad K_{G2} = \left.\frac{2}{1 + \tau_2 D}\right|_{D=0} = 2, \qquad K_H = 0.5$$

$$A = \frac{1}{K_{G1}K_{G2}} + K_H = \frac{1}{(1)(2)} + 0.5 = 1$$

Block-diagram representation of steady-state condition shown in **Figure**,



$$c = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} v + \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} u$$

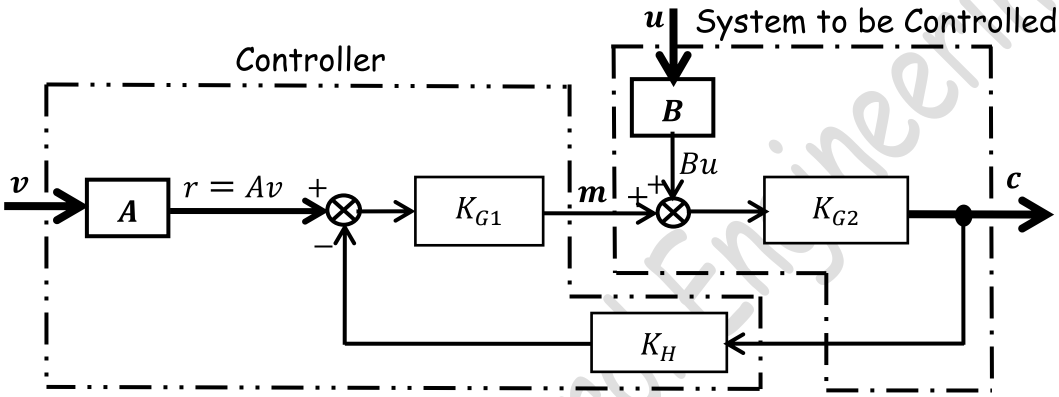
$$c = \frac{(1)(1)(2)}{1 + (1)(2)(0.5)} v + \frac{(-5)(2)}{1 + (1)(2)(0.5)} u$$

Steady-state response of control system,

$$\boldsymbol{c = v - 5u}$$

Steady-State Equation (Graphically):

Steady-state equation and steady-state constants could be determined graphically from steady-state operating curves which is obtained as follows, Consider General Steady-State Block Diagram shown in **Figure**,



1. Steady-state equation of the operating controller,

$$m = (Av - K_H c)K_{G1}, \qquad or, \qquad m = AK_{G1}v - K_{G1}K_H c$$

a) Constant command signal V or $v = 0$,

$\left.\frac{m}{c}\right|_{v=0} = \left.\frac{\Delta M}{\Delta C}\right|_V = \left.\frac{\partial M}{\partial C}\right|_V = -K_{G1}K_H$ Slope of controller operating lines

b) Constant output signal C or $c = 0$,

$\left.\frac{m}{v}\right|_{c=0} = \left.\frac{\Delta M}{\Delta V}\right|_C = \left.\frac{\partial M}{\partial V}\right|_C = AK_{G1}$ Vertical spacing between lines of constant V

c) Constant manipulating signal M or $m = 0$,

$\left.\frac{v}{c}\right|_{m=0} = \left.\frac{\Delta V}{\Delta C}\right|_M = \left.\frac{\partial V}{\partial C}\right|_M = \frac{K_H}{A}$ Horizontal spacing between lines of constant V

2. Steady-state equation of the operating system to be controlled,

$$c = (m + Bu)K_{G2}, \qquad or, \qquad c = K_{G2}m + BK_{G2}u$$

a) Constant external disturbance U or $u = 0$,

$\left.\frac{m}{c}\right|_{u=0} = \left.\frac{\Delta M}{\Delta C}\right|_U = \left.\frac{\partial M}{\partial C}\right|_U = \frac{1}{K_{G2}}$ Slope of the system to be controlled lines

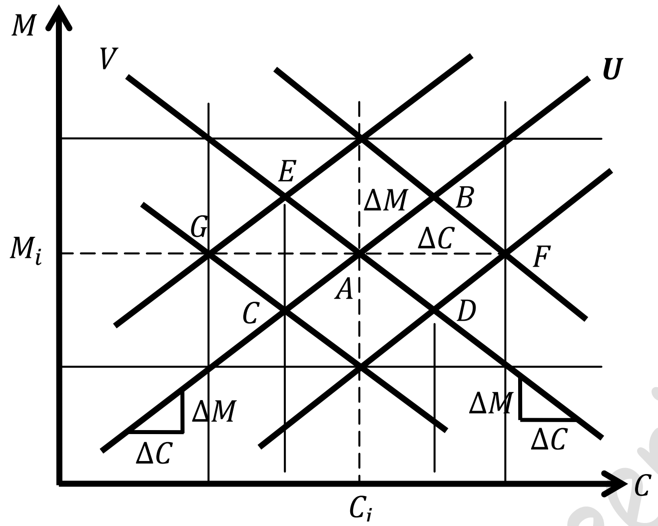
b) Constant output signal C or $c = 0$:

$\left.\frac{m}{u}\right|_{c=0} = \left.\frac{\Delta M}{\Delta U}\right|_C = \left.\frac{\partial M}{\partial U}\right|_C = -B$ Vertical spacing between lines of constant U

c) Constant manipulating signal M or $m = 0$,

$\left.\frac{c}{u}\right|_{m=0} = \left.\frac{\Delta C}{\Delta U}\right|_M = \left.\frac{\partial C}{\partial U}\right|_M = BK_{G2}$ Horizontal spacing between lines of constant U

Steady-state operating curves for controller and system to be controlled are sketched as shown in **Figure**,



Also steady-state equation is directly determined graphically from the steady-state curves,

$$C = C(V, U)$$

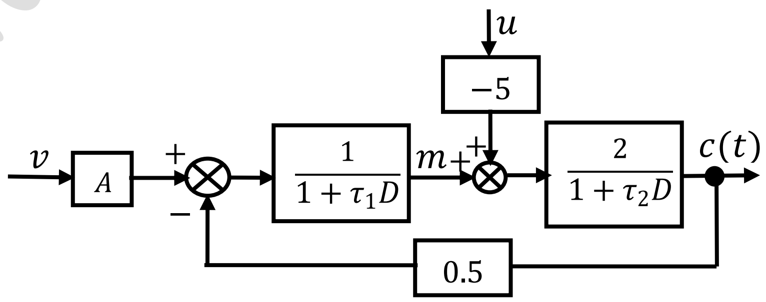
Linearization,

$$c = \left.\frac{\partial C}{\partial V}\right|_U v + \left.\frac{\partial C}{\partial U}\right|_V u$$

$$\left.\frac{\partial C}{\partial V}\right|_U = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} \qquad \left.\frac{\partial C}{\partial U}\right|_V = \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H}$$

Example 2:

Reference operating point of a control system shown in **Figure** is $V_i = C_i = 100$, $M_i = 50$, and $U_i = 10$. Determine the steady-state equation algebraically and sketch steady state operating curves. Then determine graphically the equation for steady state operation. Select constant A such that the coefficient of the desired input v is unity.



Solution: Steady-state constants are,

$$K_{G1} = \left.\frac{1}{1 + \tau_1 D}\right|_{D=0} = 1, \qquad K_{G2} = \left.\frac{2}{1 + \tau_2 D}\right|_{D=0} = 2, \qquad K_H = 0.5$$

$$A = \frac{1}{K_{G1}K_{G2}} + K_H = \frac{1}{(1)(2)} + 0.5 = 1$$

The equation of the steady-state operation (algebraically),

$$c = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} v + \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} u$$

$$c = \frac{(1)(1)(2)}{1 + (1)(2)(0.5)} v + \frac{(-5)(2)}{1 + (1)(2)(0.5)} u$$

$$\boldsymbol{c = v - 5u}$$

- Slope of the controller operating lines,

$$\left.\frac{m}{c}\right|_{v=0} = \left.\frac{\partial M}{\partial C}\right|_{V} = \left.\frac{\Delta M}{\Delta C}\right|_{V=100} = -K_{G1}K_H = -(1)(0.5) = -0.5$$

- Vertical spacing between lines of constant V

$$\left.\frac{m}{v}\right|_{c=0} = \left.\frac{\partial M}{\partial V}\right|_{C} = \left.\frac{\Delta M}{\Delta V}\right|_{C=100} = AK_{G1} = (1)(1) = 1$$

Select $\Delta M = 10$ then, $\left.\frac{\Delta M}{\Delta V}\right|_{C=100} = 1$ $\Delta M = \Delta V = 10$

- Horizontal spacing between lines of constant V

$$\left.\frac{v}{c}\right|_{m=0} = \left.\frac{\partial V}{\partial C}\right|_{M} = \left.\frac{\Delta V}{\Delta C}\right|_{M=50} = \frac{K_H}{A} = \frac{0.5}{1} = 0.5$$

$\left.\frac{\Delta V}{\Delta C}\right|_{M=50} = 0.5$, Since, $\Delta V = 10$ then, $\Delta C = 20$

- Slope of the system to be controlled operating lines,

$$\left.\frac{m}{c}\right|_{u=0} = \left.\frac{\partial M}{\partial C}\right|_{U} = \left.\frac{\Delta M}{\Delta C}\right|_{U=10} = \frac{1}{K_{G2}} = \frac{1}{2} = 0.5$$

- Vertical spacing between lines of constant U,

$$\left.\frac{m}{u}\right|_{c=0} = \left.\frac{\partial M}{\partial U}\right|_{C} = \left.\frac{\Delta M}{\Delta U}\right|_{C=100} = -B = -(-5) = 5$$

$\left.\frac{\Delta M}{\Delta U}\right|_{C=100} = 5$, Since, $\Delta M = 10$, Then, $\Delta U = 2$

- Horizontal spacing between lines of constant U,

$$\left.\frac{c}{u}\right|_{m=0} = \left.\frac{\partial C}{\partial U}\right|_{M} = \left.\frac{\Delta C}{\Delta U}\right|_{M=50} = BK_{G2} = -5(2) = -10$$

$\left.\frac{\Delta C}{\Delta U}\right|_{M=50} = -10$, Since, $\Delta U = 2$, Then, $\Delta C = -20$

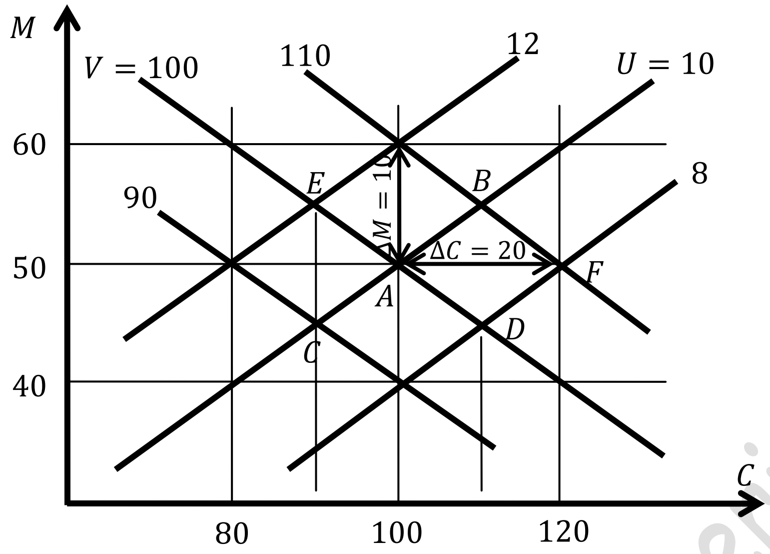
Steady-state operating curves shown in **Figure** and,

$$c = \left.\frac{\partial C}{\partial V}\right|_{U} v + \left.\frac{\partial C}{\partial U}\right|_{V} u$$

$$\left.\frac{\partial C}{\partial V}\right|_{U} = \left.\frac{\Delta C}{\Delta V}\right|_{U=10} = \frac{C_B - C_C}{V_B - V_C} = \frac{110 - 90}{110 - 90} = 1$$

$$\left.\frac{\partial C}{\partial U}\right|_{V} = \left.\frac{\Delta C}{\Delta U}\right|_{V=100} = \frac{C_D - C_E}{U_D - U_E} = \frac{110 - 90}{8 - 12} = -5$$

Steady-state equation, $c = v - 5u$

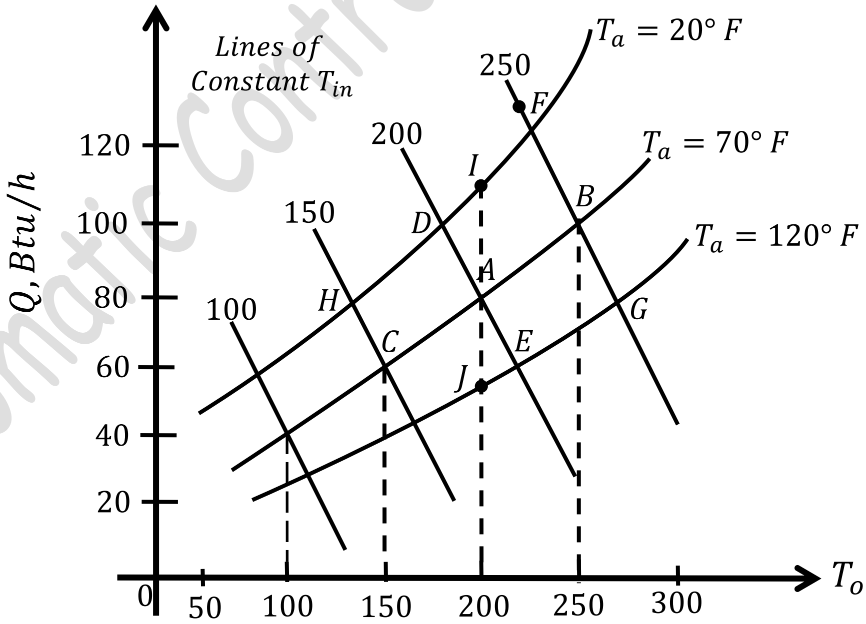


Example 3:

A typical family of steady-state operating curves for a proportional temperature control system shown in **Figure**,

a) Determine the equation for steady-state operation about point A.

b) If this were an open-loop rather than a closed-loop system, what would be the steady-state equation of operation?



Solution:

1) Since, $T_o = T_o(T_{in}, T_a)$, linearization gives,

$$t_o = \left.\frac{\partial T_o}{\partial T_{in}}\right|_{T_a} t_{in} + \left.\frac{\partial T_o}{\partial T_a}\right|_{T_{in}} t_a$$

$$\left.\frac{\partial T_o}{\partial T_{in}}\right|_{T_a} = \left.\frac{\Delta T_o}{\Delta T_{in}}\right|_{T_{a=70}} = \frac{T_{o_B} - T_{o_C}}{T_{inB} - T_{inC}} = \frac{250 - 150}{250 - 150} = 1$$

$$\left.\frac{\partial T_o}{\partial T_a}\right|_{T_{in}} = \left.\frac{\Delta T_o}{\Delta T_a}\right|_{T_{in=200}} = \frac{T_{oD} - T_{oE}}{T_{aD} - T_{aE}} = \frac{180 - 220}{20 - 120} = 0.4$$

Equation of the steady-state operation,

$$t_o = t_{in} + 0.4t_a$$

2) For open-loop control system, controller lines should be horizontal because $K_H = 0$,

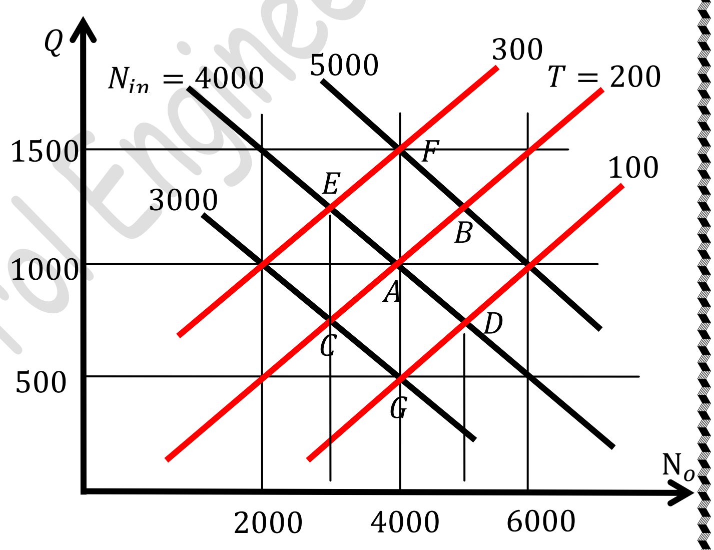
$$\left.\frac{\partial T_o}{\partial T_{in}}\right|_{T_a} = \left.\frac{\Delta T_o}{\Delta T_{in}}\right|_{T_{a=70}} = \frac{T_{oB} - T_{oC}}{T_{inB} - T_{inC}} = \frac{250 - 150}{250 - 150} = 1$$

$$\left.\frac{\partial T_o}{\partial T_a}\right|_{T_{in}} = \left.\frac{\Delta T_o}{\Delta T_a}\right|_{T_{in=200}} = \frac{T_{oG} - T_{oH}}{T_{aG} - T_{aH}} = \frac{270 - 130}{120 - 20} = 1.4$$

$$t_o = t_{in} + 1.4t_a$$

Example 4:

A typical family of steady-state operating curves for a unity feedback $(K_H = 1)$ speed control system is shown in **Figure**. At the reference operating condition (point A) $N_{in} = N_o = 4000$, $Q_i = 1000$, and $T_i = 200$.



1. Determine the steady-state constants and the equation for steady-state operation.
2. With N_{in} held fixed at its reference value, what is the change in speed N_o when the load T changes from the reference value $T_i = 200$ to 300 ?
3. By what factor should the slope of the controller lines be changed so as to reduce this change by a factor of 50 ?

Solution:

1) The steady state constants:

$$-K_{G1}K_H = \left.\frac{\partial M}{\partial C}\right|_V = \left.\frac{\Delta M}{\Delta C}\right|_V \quad \text{Slope of the controller operating lines}$$

$$-K_{G1}K_H = \left.\frac{\Delta Q}{\Delta N_o}\right|_{N_{in=4000}} = \frac{Q_D - Q_E}{N_{oD} - N_{oE}} = \frac{750 - 1250}{5000 - 3000} = -\frac{1}{4}$$

For $K_H = 1$, then $K_{G1} = \frac{1}{4}$

$$AK_{G1} = \left.\frac{\partial M}{\partial V}\right|_C = \left.\frac{\Delta M}{\Delta V}\right|_C \quad \text{Vertical spacing between lines of constant } V$$

$$AK_{G1} = \left.\frac{\Delta Q}{\Delta N_{in}}\right|_{N_o=4000} = \frac{Q_F - Q_G}{N_{inF} - N_{inG}} = \frac{1500 - 500}{5000 - 3000} = \frac{1}{2}$$

Since $K_{G1} = \frac{1}{4}$, then $A = 2$

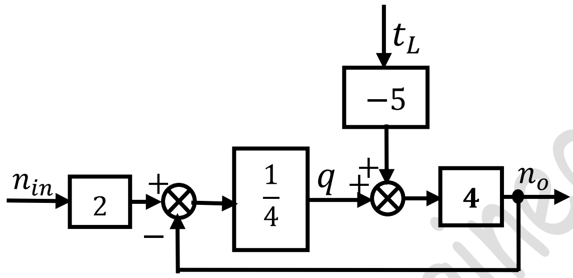
$\frac{1}{K_{G2}} = \frac{\partial M}{\partial C}\Big|_U = \frac{\Delta M}{\Delta C}\Big|_U$ Slope of the system to be controlled operating lines

$\frac{1}{K_{G2}} = \frac{\Delta Q}{\Delta N_o}\Big|_{T=200} = \frac{Q_B - Q_C}{N_{oB} - N_{oC}} = \frac{1250-750}{5000-3000} = \frac{1}{4}$, $K_{G2} = 4$

$-B = \frac{\partial M}{\partial U}\Big|_C = \frac{\Delta M}{\Delta U}\Big|_C$ Vertical spacing between lines of constant U

$-B = \frac{\Delta Q}{\Delta T}\Big|_{N_{o=4000}} = \frac{Q_F - Q_G}{T_F - T_G} = -\frac{1500-500}{300-100} = 5$, $B = -5$

Steady-state block diagram shown in **Figure**,



And equation of steady-state operation,

$$n_o = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} n_{in} + \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} t_L \qquad n_o = n_{in} - 10t_L$$

This equation also could be obtained using as:

$$c = \frac{\partial C}{\partial V}\Big|_U v + \frac{\partial C}{\partial U}\Big|_V u$$

$$n_o = \frac{\partial N_o}{\partial N_{in}}\Big|_T n_{in} + \frac{\partial N_o}{\partial T}\Big|_{N_{in}} t_L$$

$$\frac{\partial N_o}{\partial N_{in}}\Big|_T = \frac{\Delta N_o}{\Delta N_{in}}\Big|_{T=200} = \frac{N_{oB} - N_{oC}}{N_{inB} - N_{inC}}\Big|_T = \frac{5000 - 3000}{5000 - 3000} = 1$$

$$\frac{\partial N_o}{\partial T}\Big|_{N_{in}} = \frac{\Delta N_o}{\Delta T}\Big|_{N_{in=4000}} = \frac{N_{oD} - N_{oE}}{T_D - T_E}\Big|_{N_{in}} = \frac{5000 - 3000}{100 - 300} = -10$$

$$n_o = n_{in} - 1ot_L$$

2) Since N_{in} is held fixed at reference value then,

$$n_{in} = \Delta N_{in} = N_{in} - N_{ini} = 0$$

$$t_L = \Delta T = T - T_i = 300 - 200 = 100$$

Change in speed N_o ,

$$n_o = n_{in} - 10t_L = 0 - 10(100) = -1000$$

3) To decrease the change in speed by a factor of 50:

$$\frac{\partial N_o}{\partial T}\Big|_{N_{in}} = \frac{\Delta N_o}{\Delta T}\Big|_{N_{in}} = \frac{n_o}{t_L}\Big|_{n_{in=0}} = (-10)\frac{1}{50} = -\frac{1}{5}$$

Since,

$$\frac{\partial N_o}{\partial T}\Big|_{N_{in}} = \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} = -\frac{1}{5} \qquad \frac{\partial N_o}{\partial T}\Big|_{N_{in}} = \frac{(-5)(4)}{1 + 4K_{G1}K_H} = -\frac{1}{5}$$

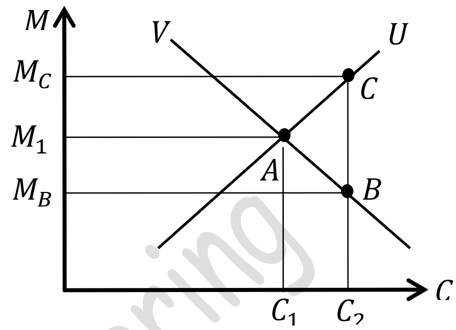
Solving for $K_{G1}K_H$

$$K_{G1}K_H = \frac{99}{4} = 99 \times \frac{1}{4}$$

Slope of the controller lines must be increased by a factor of 99

Equilibrium:

Equilibrium point is the intersection of line of command signal $\boldsymbol{V}$ for controller and load line $\boldsymbol{U}$ for the system to be controlled as shown in **Figure**. In which, manipulated variable being supplied by controller is the same that required to maintain system output at reference point. At equilibrium point A the amount of manipulated variable M_1 supplied by controller is same as that required to maintain the system output at C_1. When output is changed to C_2 which is higher than C_1 then point C decreases until equilibrium is attained at point A because $M_B \ll M_c$

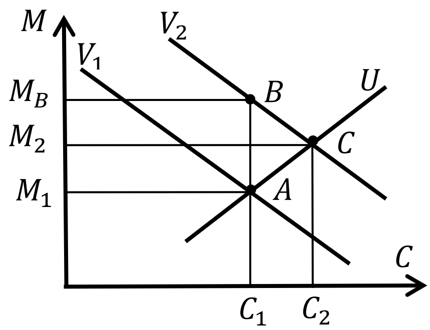


Response for changing command signal and load (Graphically):

Response of any control system for changing in command signal and load could be investigated graphically using steady-state operating curves.

1. Finite slope of controller lines:

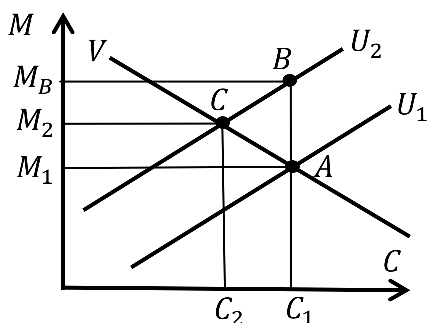
* When V is changed from V_1 to V_2 $(V_2 \gg V_1)$ with load U is constant, a new operating point for controller is at B while system to be controlled remains at A. Since $M_B \gg M_1$, output C increases (dynamically) and new equilibrium point of operation is existed at point C,



$$\left.\frac{c}{v}\right|_{u=0} = \left.\frac{\partial C}{\partial V}\right|_U = \left.\frac{\Delta C}{\Delta V}\right|_U = \frac{AK_{G1}K_{G2}}{1 + K_{G1}K_{G2}K_H} = 1$$

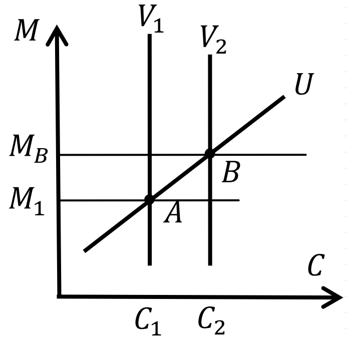
$$A = \frac{1}{K_{G1}K_{G2}} + K_H$$

* When U is changed from U_1 to U_2 $(U_2 \gg U_1)$ with constant V, a new point of operation for system to be controlled at B while the controller remains at A . Since, $M_1 \ll M_B$, output C decreases (dynamically) and new equilibrium point of operation is existed at C.



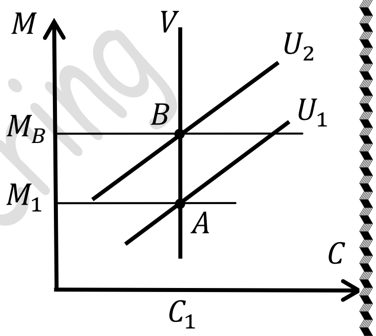
$$\left.\frac{c}{u}\right|_{v=0} = \left.\frac{\partial C}{\partial U}\right|_V = \left.\frac{\Delta C}{\Delta U}\right|_V = \frac{BK_{G2}}{1 + K_{G1}K_{G2}K_H} = \text{finite value}$$

2. Infinite slope (vertically) of controller lines:



* When V is changed from V_1 to V_2 $(V_2 \gg V_1)$ with load U is constant, a new operating point for controller and system to be controlled is at B (new equilibrium point). Manipulated variable supplied is equal to the quantity required without dynamically movement,

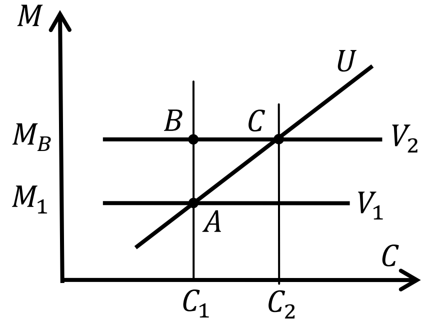
$$\left.\frac{c}{v}\right|_{u=0} = \left.\frac{\partial C}{\partial V}\right|_U = \left.\frac{\Delta C}{\Delta V}\right|_U = \frac{AK_{G1}K_{G2}}{1+K_{G1}K_{G2}K_H} = \frac{A}{K_H} = 1 \qquad A = K_H$$



* When U is changed from U_1 to U_2 $(U_2 \gg U_1)$ with constant V, a new point of operation for system to be controlled and the controller is at B (new equilibrium point). Manipulated variable supplied is equal to the quantity required without dynamically movement,

$$\left.\frac{c}{u}\right|_{v=0} = \left.\frac{\partial C}{\partial U}\right|_V = \left.\frac{\Delta C}{\Delta U}\right|_V = \frac{BK_{G2}}{1+K_{G1}K_{G2}K_H} = 0$$

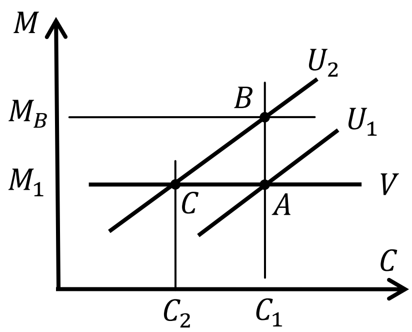
3. Zero slope (horizontally) of controller lines:



* When V is changed from V_1 to V_2 $(V_2 \gg V_1)$ with load U is constant, a new operating point for controller is at B while system to be controlled remains at A.
  Since, $M_B \gg M_1$, output C increases (dynamically) and new equilibrium point of operation is existed at point C,

$$\left.\frac{c}{v}\right|_{u=0} = \left.\frac{\partial C}{\partial V}\right|_U = \left.\frac{\Delta C}{\Delta V}\right|_U = \frac{AK_{G1}K_{G2}}{1+K_{G1}K_{G2}K_H} = AK_{G1}K_{G2} = 1$$

$$A = \frac{1}{K_{G1}K_{G2}}$$



* When U is changed from U_1 to U_2 $(U_2 \gg U_1)$ with constant V, a new point of operation for system to be controlled is at B while the controller remains at A.
  Since, $M_B \gg M_1$, output C decreases (dynamically) and new equilibrium point of operation is existed at C.

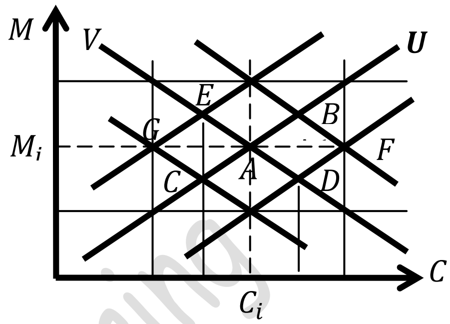
$$\left.\frac{c}{u}\right|_{v=0} = \left.\frac{\partial C}{\partial U}\right|_V = \left.\frac{\Delta C}{\Delta U}\right|_V = \frac{BK_{G2}}{1+K_{G1}K_{G2}K_H} = BK_{G2} = max.value$$

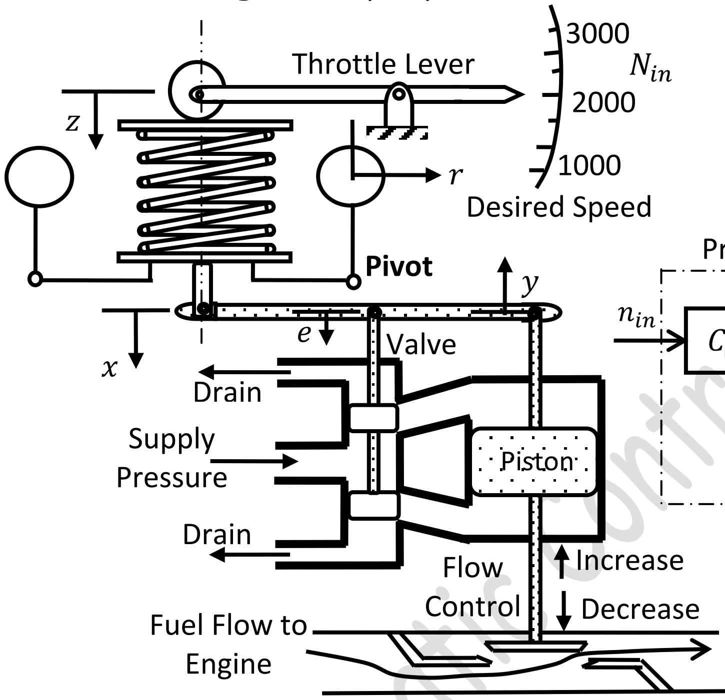
Types of Control Systems:

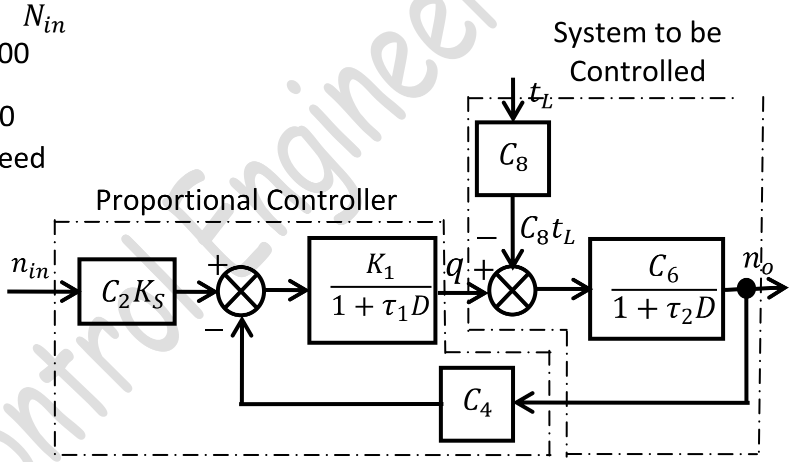
Slope of controller lines $(-K_{G1}K_H)$, varies from zero for an open loop control system $(K_H = 0)$ to infinity for an integral control system $(K_{G1} = \infty)$. A proportional control system has finite slope.

1. Proportional Control Systems (P Controller):

In proportional control systems the coefficient of external disturbance (load) u term is finite with finite slope of controller lines as shown in **Figure**. Speed control of gas turbine for jet plane shown in **Figure** is proportional control system.





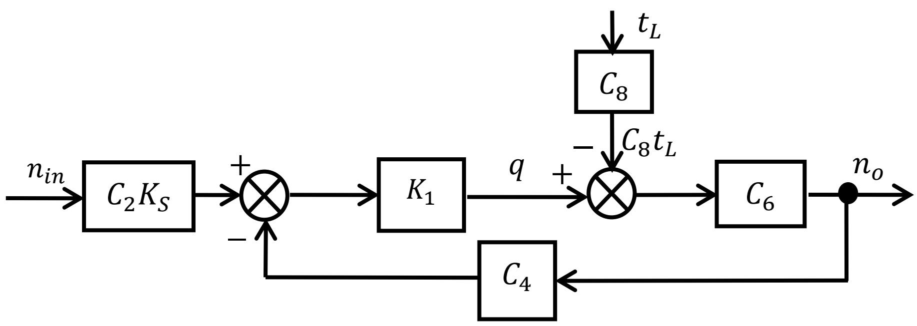


$$n_o = \frac{C_2K_sK_1C_6}{K_1C_6C_4 + (1+\tau_1 D)(1+\tau_2 D)} n_{in} - \frac{C_8C_6(1+\tau_1 D)}{K_1C_6C_4 + (1+\tau_1 D)(1+\tau_2 D)} t_L$$

Mathematical equation for steady-state operation is obtained by letting $D = 0$:

$$n_o = \frac{C_2K_sK_1C_6}{1+K_1C_6C_4} n_{in} - \frac{C_8C_6}{1+K_1C_6C_4} t_L$$

Also steady-state overall block diagram representation is obtained,



Since $K_{G1} = K_1$ and $K_H = C_4$, then slope of the controller lines is finite and it is proportional control system.

Since $K_{G2} = C_6$,

$$A = \frac{1}{K_{G1}K_{G2}} + K_H = \frac{1}{K_1C_6} + C_4 = C_2K_s \qquad C_2 = \frac{1}{K_s}\left[\frac{1}{K_1C_6} + C_4\right]$$

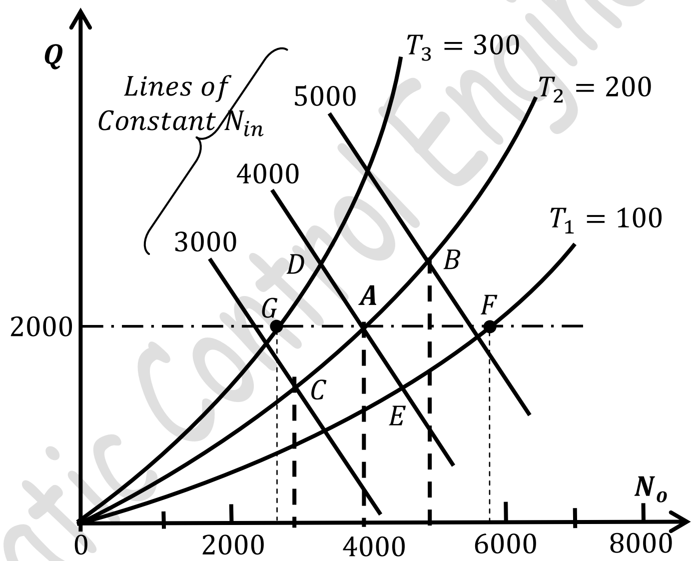
The term $C_2 = \partial Z/\partial N_{in}|_i$ is the scale factor for the speed setting dial,

Example 5:

A typical family of steady-state operating curves for a proportional speed control system of a diesel or turbine shown in **Figure**. Determine:

a) Steady-state equation of operation in the vicinity of point A.

b) If this were an open-loop rather than a closed-loop system, what would be the steady-state equation of operation about point A.



Solution:

a) Since, $N_o = N_o(N_{in}, T)$

$$n_o = \left.\frac{\partial N_o}{\partial N_{in}}\right|_T n_{in} + \left.\frac{\partial N_o}{\partial T}\right|_{N_{in}} t_L$$

$$\left.\frac{\partial N_o}{\partial N_{in}}\right|_{T_2=200} = \left.\frac{\Delta N_o}{\Delta N_{in}}\right|_{T_2=200} = \frac{N_{oB} - N_{oC}}{N_{inB} - N_{inC}} = \frac{5000 - 3000}{5000 - 3000} = 1$$

$$\left.\frac{\partial N_o}{\partial T}\right|_{N_{in=4000}} = \left.\frac{\Delta N_o}{\Delta T}\right|_{N_{in=4000}} = \frac{N_{oD} - N_{oE}}{T_D - T_E} = \frac{3400 - 4500}{300 - 100} = \frac{-1100}{200} = -5.5$$

$$n_o = n_{in} - 5.5t_L$$

b) Controller lines of open-loop control system are horizontal because $K_H = 0$,

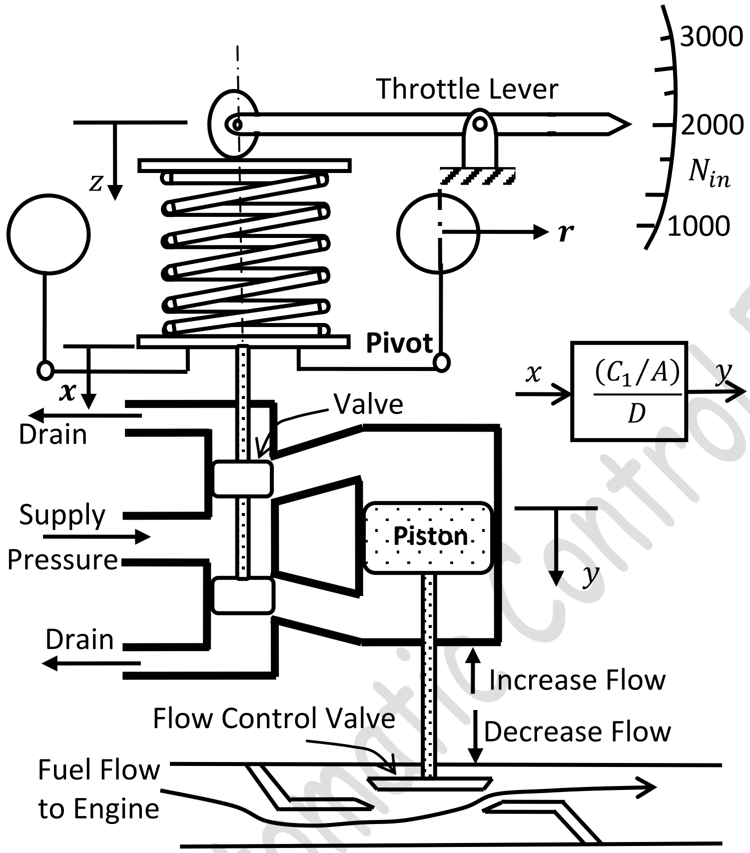
$$\left.\frac{\partial N_o}{\partial N_{in}}\right|_{T_2=200} = \left.\frac{\Delta N_o}{\Delta N_{in}}\right|_{T_2=200} = \frac{N_{oB} - N_{oC}}{N_{inB} - N_{inC}} = \frac{5000 - 3000}{5000 - 3000} = 1$$

$$\left.\frac{\partial N_o}{\partial T}\right|_{N_{in}=4000} = \left.\frac{\Delta N_o}{\Delta T}\right|_{N_{in}=4000} = \frac{N_{oF} - N_{oG}}{T_F - T_G} = \frac{5800 - 2700}{100 - 300} = \frac{3100}{-200} = -15.5$$

$$n_o = n_{in} - 15.5t_L$$

2. Integral Control Systems:

Integral control system has an integrating component yields $(1/D)$ term therefore their controller lines of steady-state operating curves have an infinite slope (vertically). Proportional controller could be inverted to an integral controller by eliminating the walking beam linkage and using hydraulic integrator as shown in **Figure**.



$Q = Q(X, P)$, Linearization,

$$q = C_1x - C_2p$$

Then, $q = C_1x - \frac{C_2M}{A}D^2y$

$$ADy = C_1x - \frac{C_2M}{A}D^2y$$

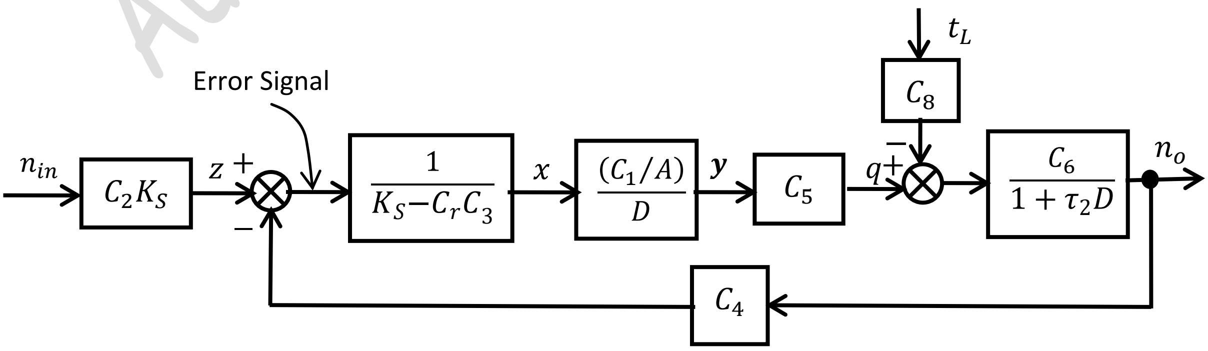
Let $\tau = CM/A^2$

$$y = \frac{(C_1/A)}{D(1 + \tau D)}x$$

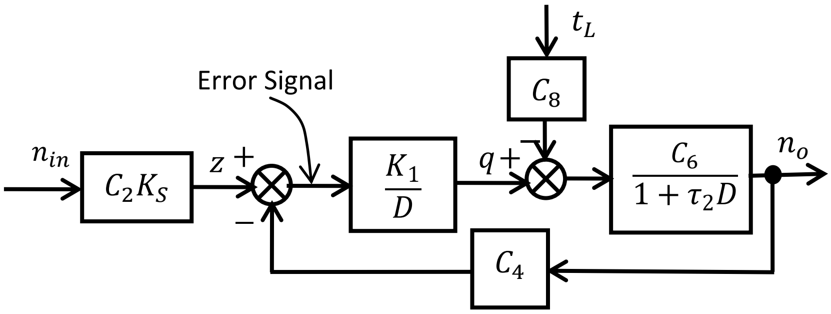
For negligible load, $\tau = 0$

$$y = \frac{(C_1/A)}{D}x$$

Overall block diagram representation for speed control system is:



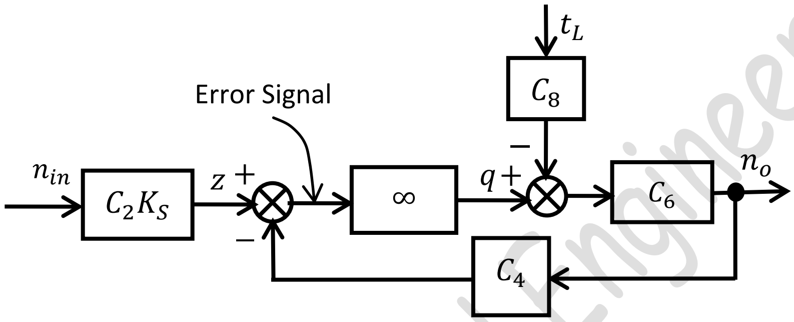
Using the rule of combining blocks in cascade and let $K_1 = \frac{C_5(C_1/A)}{(K_S - C_rC_3)}$



Mathematical differential equation of operation,

$$n_o = \frac{C_2K_sK_1C_6}{K_1C_6C_4 + D(1+\tau_2 D)}n_{in} - \frac{C_8C_6D}{K_1C_6C_4 + D(1+\tau_2 D)}t_L$$

Steady-state block diagram representation could be obtained by letting $= 0$,



Also, steady-state equation,

$$n_o = \frac{C_2K_s}{C_4}n_{in}$$

Since $K_{G1} = \infty$ and $K_H = C_4$, then slope of controller lines is infinite (vertical lines) and this speed control system is considered as an integral control system. From the integral element K_1/D , the steady-state operation constant is:

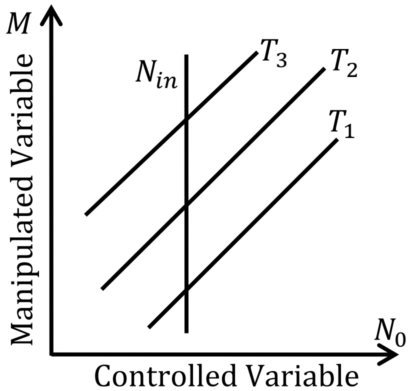
$$K_{G1} = \left(\frac{K_1}{D}\right)_{D=0} = \infty, \quad \text{since,} \qquad q = \left(\frac{K_1}{D}\right)_{D=0} e$$

$$q = K_{G1}e \quad \text{or,} \qquad q = \infty * e \qquad\qquad e = 0$$

So, no steady state error in integral control system, then

$$C_2K_sn_{in} - C_4n_o = e = 0$$

$$n_o = \frac{C_2K_s}{C_4}n_{in}$$

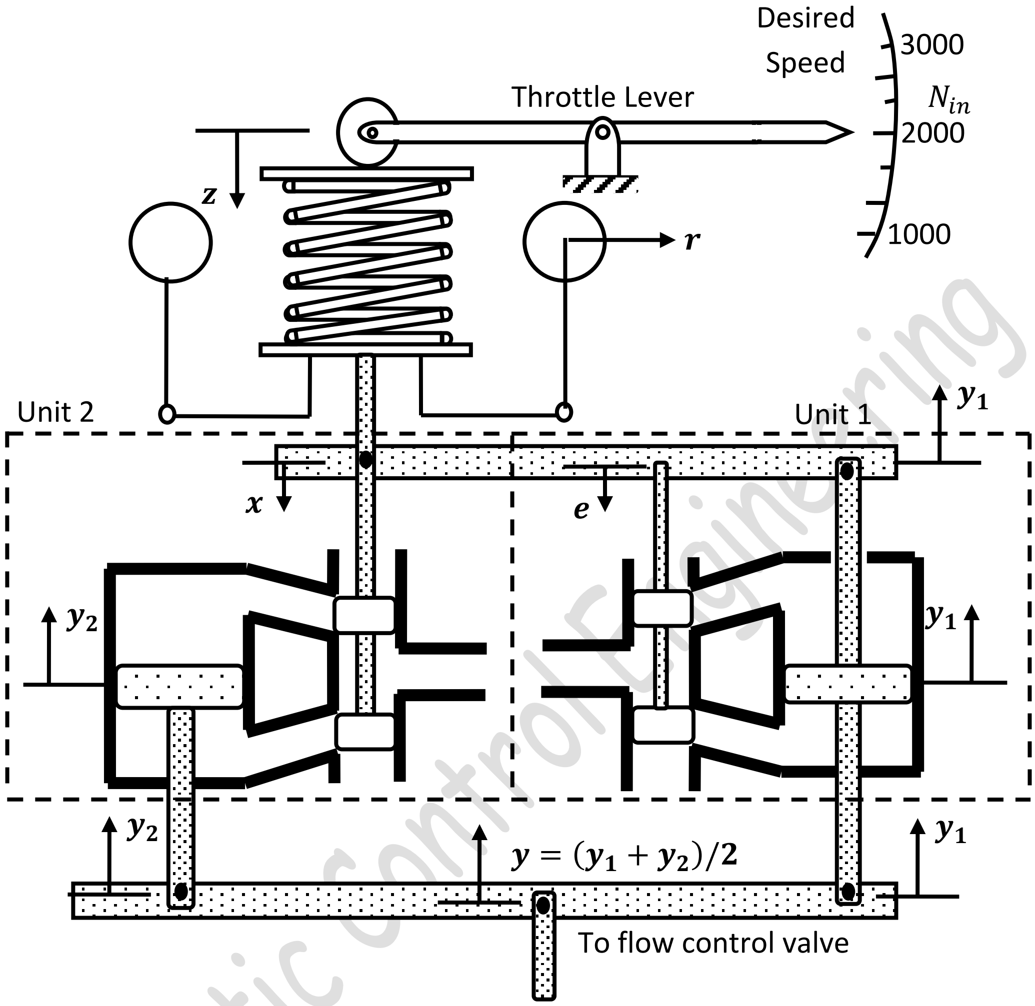


Speed (output variable) is independent of change in load torque (external disturbance variable) for an integral control system as shown **in Figure**. It is an easy matter to adjust the scale factor C_2 for the input speed dial so that the coefficient of the desired input is unity $\frac{C_2K_S}{C_4} = 1$,

$$n_o = n_{in}$$

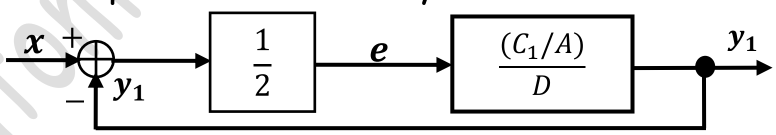
3. Proportional Plus Integral Control Systems:

A proportional plus integral controller as shown in **Figure** combines desirable transient characteristics of a proportional controller and the feature of no steady-state error of the integral controller.

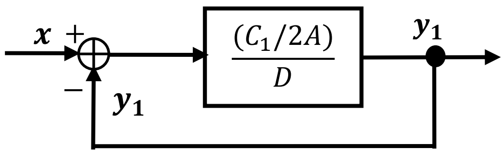


- Proportional action is provided by unit 1,

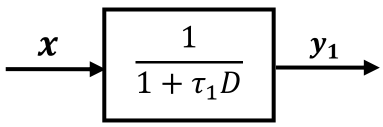
Block Diagram Representation of Hydraulic Servomotors,



Using the rule of combining blocks in cascade yields,



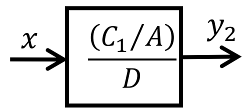
Using the rule of eliminating feedback loop yields and let, $\frac{2A}{C_1} = \tau_1$,



Then proportional equation is,

$$y_1 = \frac{1}{1+\tau_1 D}x$$

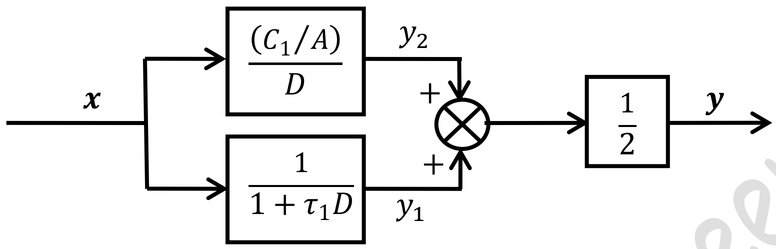
❖ Integral action is provided by unit 2,



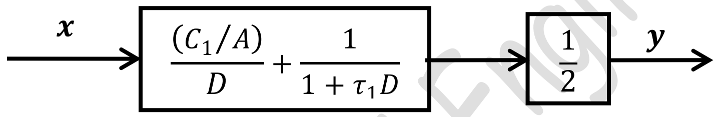
$$y_2 = \frac{(C_1/A)}{D}x \qquad y = \frac{y_1 + y_2}{2}$$

$$y = \frac{1}{2}\left(\frac{1}{1+\tau_1 D} + \frac{(C_1/A)}{D}\right)x$$

This equation could be represented by a block diagram,

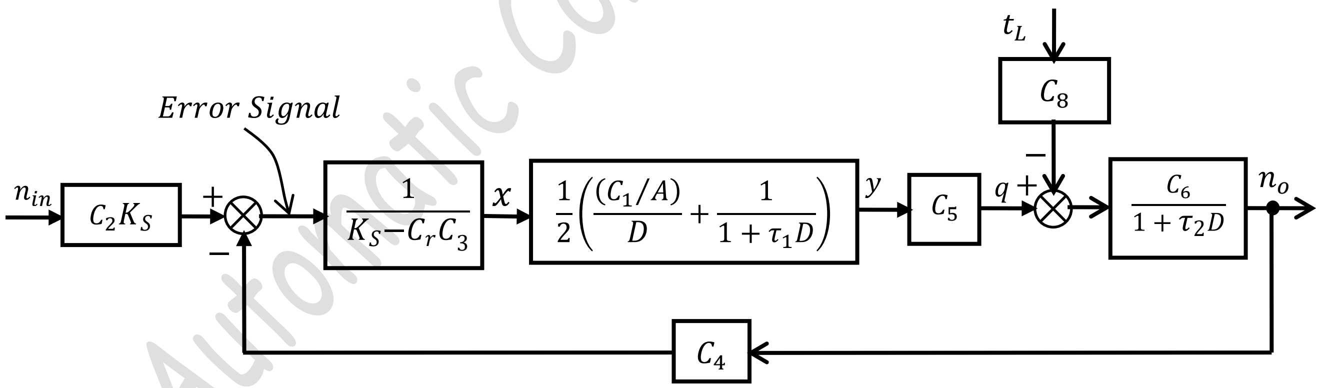


Using the rule of combining blocks in parallel,

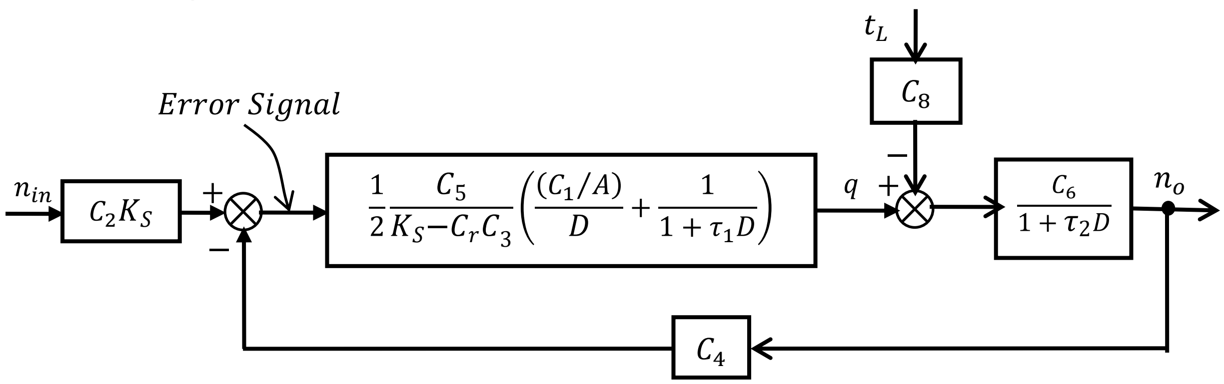


Using the rule of combining blocks in cascade,

Overall block diagram for the proportional plus integral control system,



Using the rule of combining blocks in cascade,



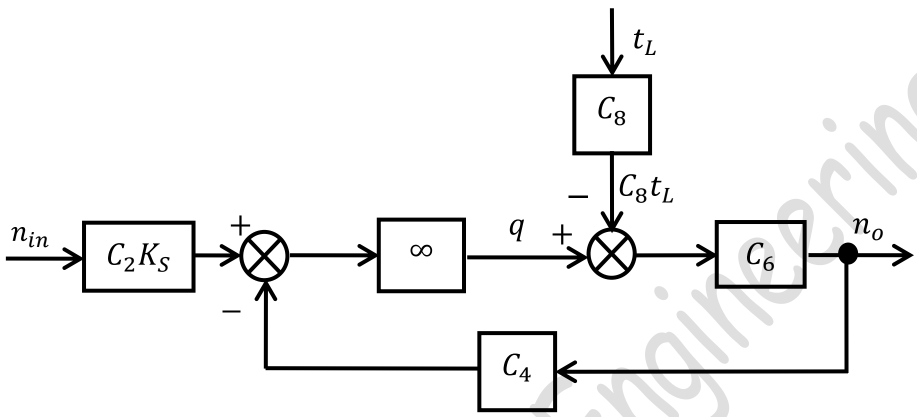
The steady –state constants could be evaluated as,

$$K_{G1} = [G_1(D)]_{D=0} = \left[\frac{1}{2}\frac{C_5}{K_S - C_r C_3}\left(\frac{(C_1/A)}{D} + \frac{1}{1+\tau_1 D}\right)\right]_{D=0} = \infty$$

$$K_{G2} = [G_2(D)]_{D=0} = \left[\frac{C_6}{1+\tau_2 D}\right]_{D=0} = C_6$$

$$K_H = [H(D)]_{D=0} = C_4$$

And steady-state block diagram representation could be obtained by letting $= 0$,



Steady-state equation,

$$n_o = \frac{A K_{G1} K_{G2}}{1 + K_{G1} K_{G2} K_H} n_{in} + \frac{B K_{G2}}{1 + K_{G1} K_{G2} K_H} t_L$$

$$n_o = \frac{C_2 K_s}{C_4} n_{in} = n_{in}$$

This equation refers that the steady-state operation of proportional plus integral control system is the same as of an integral control system alone in which it follows that e is zero during steady-state operation,

$$C_2 K_s n_{in} - C_{4 n_o} = e = 0$$

Also,

$$n_o = \frac{C_2 K_s}{C_4} n_{in} = n_{in}$$

Chapter Five

Laplace Transforms

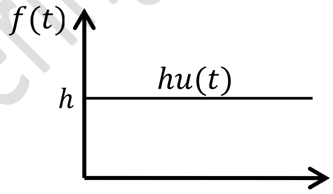
In the method of Laplace transformation, differential equations are transformed into algebraic equations with variable s while time solution or transient response is obtained by Laplace Inverse. Laplace transformation $F(s)$ of a function of time $f(t)$ is,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Hint:
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

1. Step function:

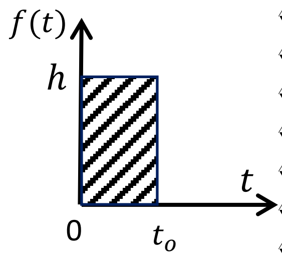
Step function shown in **Figure,** is $hu(t)$ where h is height while $u(t)$ is a unit step function whose height is one. Laplace transform of step function,



$$F(s) = \mathcal{L}[hu(t)] = \int_0^\infty hu(t)e^{-st}dt = -\frac{he^{-st}}{s}\bigg|_0^\infty = \frac{h}{s}$$

2. Pulse function:

Pulse function as shown in **Figure**.



$$f(t) = \begin{cases} h & 0 \le t \le t_o \\ 0 & t > t_o \end{cases}$$

Laplace transform of pulse function is,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^{t_o} he^{-st}dt + \int_{t_o}^\infty (0)e^{-st}dt$$

$$F(s) = \int_0^{t_o} he^{-st}dt = h\left(\frac{-e^{-st}}{s}\right)\bigg|_0^{t_o} = \frac{h}{s}(1 - e^{-st_o})$$

3. Impulse function:

An impulse function is a limited case of pulse function in which t_o approaches to zero. It is designated by $ku_1(t)$ whose area is $k = ht_o$. Where, $u_1(t)$ is unit impulse function whose area is unity. Since Laplace transform of pulse function is,

$$F(s) = \frac{h}{s}(1 - e^{-st_o})$$

Laplace transform of impulse function where $h = \frac{k}{t_o}$,

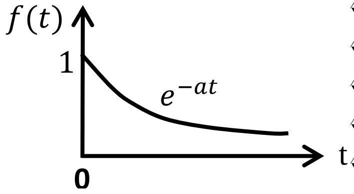
Hint:
$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$F(s) = \lim_{t_o \to 0}\left[\frac{k}{t_o s}(1 - e^{-st_o})\right]$$

Which is impossible but it could be obtained using L'hopital rule,

$$F(s) = \lim_{t_o \to 0}\left[\frac{k}{t_o s}(1 - e^{-st_o})\right] = \lim_{t_o \to 0}\frac{(d/dt_o)[k(1 - e^{-st_o})]}{(d/dt_o)(t_o s)} = \frac{kse^{-st_o}}{s} = k$$

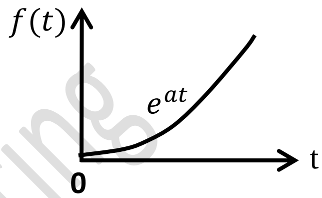
4. Exponentially decaying function:



Exponentially decaying function $f(t) = e^{-at}$ is shown in **Figure**. Laplace transform of exponentially decaying function is,

$$F(s) = \mathcal{L}(e^{-at}) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = -\frac{e^{-(s+a)t}}{s+a}\bigg|_0^\infty = \frac{1}{s+a}$$

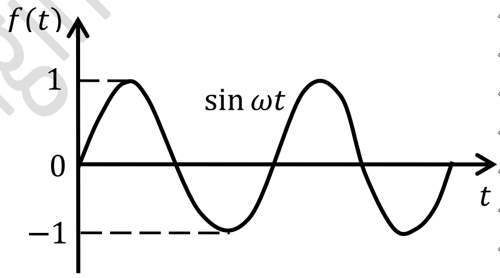
5. Exponentially Increasing function:



Exponentially increasing function $f(t) = e^{at}$ is shown in **Figure**, Laplace transform of exponentially decaying function is,

$$F(s) = \mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = -\frac{e^{-(s-a)t}}{s-a}\bigg|_0^\infty = \frac{1}{s-a}$$

6. Sinusoidal Function:



A sinusoidal function shown in **Figure** is expressed as, $f(t) = \sin \omega t$

> Hint:
> $$\int e^{ax}\sin bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx)$$

Laplace transform of sinusoidal function,

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty \sin \omega t\, e^{-st}dt = \frac{\omega}{s^2+\omega^2}$$

Table 5-1 Laplace transforms of most functions arise in control problems.

Type of Function	$f(t)$	$F(s)$	Type of Function	$f(t)$	$F(s)$
Step Function	$hu(t)$	$\frac{h}{s}$	exponentially increasing function	e^{at}	$\frac{1}{s-a}$
Unit Step Function	$u(t)$	$\frac{1}{s}$		$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
Impulse Function	$ku_1(t)$	k	Sinusoidal Function	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Unit Impulse Function	$u_1(t)$	1	Sinusoidal Function	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
unit ramp Function	t	$\frac{1}{s^2}$	Sinusoidal Function	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
Exponential Function	t^n	$\frac{n!}{s^{n+1}}$	Sinusoidal Function	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
exponentially decaying function	e^{-at}	$\frac{1}{s+a}$			

Laplace Transform Properties:

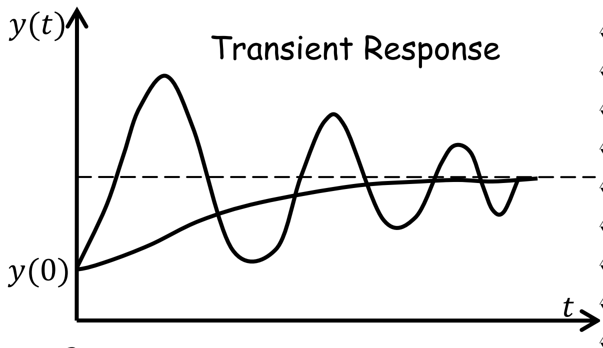
1. $\mathcal{L}[kf(t)] = kF(s)$
2. $\mathcal{L}[f_1(t) \mp f_2(t)] = F_1(s) \mp F_2(s)$
3. $\mathcal{L}[Df(t)] = sF(s) - f(0)$
4. $\mathcal{L}[D^2 f(t)] = s\mathcal{L}[Df(t)] - Df(0)$
   $= s[sF(s) - f(0)] - Df(0) = s^2F(s) - sf(0) - f'(0)$
5. $\mathcal{L}[D^3 f(t)] = s\mathcal{L}[D^2 f(t)] - D^2 f(0) = s[s^2F(s) - sf(0) - Df(0)] - D^2 f(0)$
   $= s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$
6. $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$
7. $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$
8. $\mathcal{L}[\int f(t)\,dt] = \frac{\int f(t)dt}{s}\Big|_{t=0} + \frac{1}{s}F(s)$

Laplace Transform Theorems:

1. Initial-Value Theorem:
   Initial-value theorem is used to determine the value $f(t)$ at zero time from Laplace transform $F(s)$;
   $$f(0) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$
2. Final-Value Theorem:
   Final-value theorem is used to determine the value $f(t)$ at infinity ∞ from Laplace transform $F(s)$;
   $$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Transient Response:

Transient response means that a control system changes from some initial operating condition to final condition as shown in **Figure**. Transient responses or the time solution could be determined by solving the general mathematical differential equation,



$$y(t) = \frac{a_m D^m + a_{m-1}D^{m-1} + a_{m-2}D^{m-2} + \cdots + a_1 D + a_0}{D^n + b_{n-1}D^{n-1} + b_{n-2}D^{n-2} + \cdots + b_1 D + b_0} f(t)$$

$a_0, a_1, \ldots, a_m,\ b_0, b_1, \ldots, b_n$ are constants and,

$y(t)$ Transient response function or output function

$f(t)$ Forcing function or excitation function (input function)

Let,

$$L_m(D) = a_m D^m + a_{m-1}D^{m-1} + a_{m-2}D^{m-2} + \cdots + a_1 D + a_0$$

$$L_n(D) = D^n + b_{n-1}D^{n-1} + b_{n-2}D^{n-2} + \cdots + b_1 D + b_0$$

$L_n(D)$ Characteristic function

$L_n(D) = 0$ Characteristic equation

Exponent n is the highest power of D in the characteristic function,

$$y(t) = \frac{L_m(D)}{L_n(D)} f(t) \qquad Where, \frac{L_m(D)}{L_n(D)} \text{ is the transfer function}$$

Different methods are used to solve this equation and obtain time solution.

<u>Solution with Laplace Transform Method:</u>

General mathematical differential equation of operation,

$$y(t) = \frac{L_m(D)}{L_n(D)} f(t)$$

Laplace Transform,

$$Y(s) = \frac{L_m(s)F(s) + I(s)}{L_n(s)}$$

Since, $F(s) = \frac{N_{F(s)}}{D_{F(s)}}$

$$Y(s) = \frac{L_m(s)N_{F(s)} + I(s)D_{F(s)}}{L_n(s)D_{F(s)}} = \frac{A(s)}{B(s)}$$

Where, $A(s)$ and $B(s)$ are polynomials in s. $B(s)$ may be factored as,

$$B(s) = (s - r_1)(s - r_2)\cdots\cdots(s - r_n)$$

$$Y(s) = \frac{A(s)}{(s - r_1)(s - r_2)\cdots\cdots(s - r_n)}$$

Transient response $y(t)$ is obtained by solving the general transformed equation $Y(s)$ where zeros of polynomial $B(s)$ may be distinct or repeated,

1. **<u>Distinct Zeros:</u>**

$$r_1 \neq r_2 \neq r_3 \neq \cdots \neq r_n$$

Techniques of Partial Fraction Expansion,

$$Y(s) = \frac{K_1}{s - r_1} + \frac{K_2}{s - r_2} + \cdots + \frac{K_n}{s - r_n}$$

Laplace inverse gives the transient response of a control system,

$$y(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} \cdots\cdots + K_n e^{r_n t}$$

Where constants K_i for distinct zeros may be evaluated as,

$$K_i = \lim_{s \to r_i}\left[(s - r_i)\frac{A(s)}{B(s)}\right] = \lim_{s \to r_i}[(s - r_i)Y(s)]$$

2. **<u>Repeated Zeros:</u>**

When transformed function $B(s)$ has a repeated zero r occurs q times:

$$B(s) = (s - r)^q (s - r_1)(s - r_2)\cdots\cdots(s - r_{n-q})$$

$$Y(s) = \frac{A(s)}{(s - r)^q (s - r_1)(s - r_2)\cdots\cdots(s - r_{n-q})}$$

Techniques of Partial Fraction Expansion,

$$Y(s) = \frac{C_q}{(s-r)^q} + \frac{C_{q-1}}{(s-r)^{q-1}} + \cdots + \frac{C_1}{s-r} + \frac{K_1}{s-r_1} + \cdots + \frac{K_{n-q}}{s-r_{n-q}}$$

Laplace inverse yields the transient response of the control system:

$$y(t) = \left[\frac{C_q t^{q-1}}{(q-1)!} + \frac{C_{q-1} t^{q-2}}{(q-2)!} + \cdots + \frac{C_2 t}{1!} + C_1\right] e^{rt} + K_1 e^{r_1 t} + \cdots + K_{n-q} e^{r_{n-q} t}$$

Where constants for repeated zeros may be evaluated as:

$$C_q = \lim_{s\to r}\left[(s-r)^q \frac{A(s)}{B(s)}\right] = \lim_{s\to r}[(s-r)^q Y(s)]$$

$$C_{q-1} = \lim_{s\to r}\left\{\frac{1}{1!}\frac{d}{ds}\left[(s-r)^q \frac{A(s)}{B(s)}\right]\right\} = \lim_{s\to r}\left\{\frac{1}{1!}\frac{d}{ds}[(s-r)^q Y(s)]\right\}$$

$$C_{q-k} = \lim_{s\to r}\left\{\frac{1}{k!}\frac{d^k}{ds^k}\left[(s-r)^q \frac{A(s)}{B(s)}\right]\right\} = \lim_{s\to r}\left\{\frac{1}{k!}\frac{d^k}{ds^k}[(s-r)^q Y(s)]\right\}$$

Example 1:

Determine the transient response or the time solution of the following differential equation when all initial conditions are zero,

$$y(t) = \frac{D+4}{D^2+5D+6} f(t)$$

The forcing function is an exponentially decaying $f(t) = 2e^{-t}$. Also using Initial and Final value Theorems to determine value of transient response at zero time $y(0)$ and value of steady-state response $y(\infty)$

Solution:

$$(D^2+5D+6)y(t) = (D+4)f(t)$$

$$D^2 y(t) + 5Dy(t) + 6y(t) = Df(t) + 4f(t)$$

Laplace transforms,

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 6Y(s) = [sF(s) - f(0)] + 4F(s)$$

Since all the initial conditions are zero:

$$s^2Y(s) + 5sY(s) + 6Y(s) = sF(s) + 4F(s)$$

$$(s^2+5s+6)Y(s) = (s+4)F(s) \qquad Y(s) = \frac{s+4}{s^2+5s+6}F(s)$$

$$s^2+5s+6 = (s+2)(s+3) = 0$$ Characteristic equation

$$Y(s) = \frac{s+4}{(s+2)(s+3)}F(S)$$

$$f(t) = 2e^{-t} \qquad F(s) = \mathcal{L}[f(t)] = \mathcal{L}[2e^{-t}] = 2\mathcal{L}[e^{-t}] = \frac{2}{s+1}$$

Laplace transformed equation,

$$Y(s) = \frac{2(s+4)}{(s+1)(s+2)(s+3)}$$

Techniques of Partial-Fraction-Expansion,

$$Y(s)=\frac{K_1}{s+1}+\frac{K_2}{s+2}+\frac{K_3}{s+3} \qquad K_i=\lim_{s\to r_i}[(s-r_i)Y(s)]$$

$$K_1=\lim_{s\to -1}\left[(s+1)\frac{2(s+4)}{(s+1)(s+2)(s+3)}\right]=3$$

$$K_2=\lim_{s\to -2}\left[(s+2)\frac{2(s+4)}{(s+1)(s+2)(s+3)}\right]=-4$$

$$K_3=\lim_{s\to -3}\left[(s+3)\frac{2(s+4)}{(s+1)(s+2)(s+3)}\right]=1$$

$$Y(s)=\frac{3}{s+1}-\frac{4}{s+2}+\frac{1}{s+3}$$

Inverse Laplace gives transient response or time solution,

$$y(t)=3e^{-t}-4e^{-2t}+e^{-3t}$$

Using Initial -Value Theorem,

$$y(0)=\lim_{t\to 0}y(t)=\lim_{s\to\infty}sY(s)=\lim_{s\to\infty}s\frac{2(s+4)}{(s+1)(s+2)(s+3)}=0$$

Using Final -Value Theorem,

$$y(\infty)=\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}s\frac{2(s+4)}{(s+1)(s+2)(s+3)}=0$$

Verification,

$$y(t)=3e^{-t}-4e^{-2t}+e^{-3t}$$

Value of transient response at zero time,

$$y(0)=3e^{-(0)}-4e^{-2(0)}+e^{-3(0)}=0$$

Value of steady-state response as time approaches infinity,

$$y(\infty)=3e^{-(\infty)}-4e^{-2(\infty)}+e^{-3(\infty)}=0$$

Example 2:

Determine the transient response or the time solution of the system described by the following differential equation,

$$y(t)=\frac{6(D+2)}{D^2+7D+12}f(t)$$

Forcing function $f(t)$ is a unit step function, and all initial conditions are zero. Also using Initial and Final value Theorems to determine value of transient response at zero time $y(0)$ and value of steady-state response $y(\infty)$

Solution:

$$(D^2+7D+12)y(t)=6(D+2)f(t)$$

$$D^2y(t)+7Dy(t)+12y(t)=6Df(t)+12f(t)$$

Laplace transforms,

$$[s^2Y(s) - sy(0) - y'(0)] + 7[sY(s) - y(0)] + 12Y(s) = 6[sF(s) - f(0)] + 12F(s)$$

Since all the initial conditions are zero:

$$s^2Y(s) + 7sY(s) + 12Y(s) = 6sF(s) + 12F(s)$$

$$(s^2 + 7s + 12)Y(s) = 6(s+2)F(s)$$

$$Y(s) = \frac{6(s+2)}{s^2 + 7s + 12}F(s)$$

$s^2 + 7s + 12 = (s+3)(s+4) = 0$ Characteristic equation

$$Y(s) = \frac{6(s+2)}{(s+3)(s+4)}F(S)$$

Since $f(t)$ is a unit step function, $F(s) = 1/s$, Laplace transformed equation,

$$Y(s) = \frac{6(s+2)}{s(s+3)(s+4)}$$

Techniques of Partial-Fraction-Expansion,

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{s+4} \qquad K_i = \lim_{s \to r_i}[(s - r_i)Y(s)]$$

$$K_1 = \lim_{s \to 0}\left[(s+0)\frac{6(s+2)}{s(s+3)(s+4)}\right] = 1$$

$$K_2 = \lim_{s \to -3}\left[(s+3)\frac{6(s+2)}{s(s+3)(s+4)}\right] = 2$$

$$K_3 = \lim_{s \to -4}\left[(s+4)\frac{6(s+2)}{s(s+3)(s+4)}\right] = -3$$

$$Y(s) = \frac{1}{s} + \frac{2}{s+3} - \frac{3}{s+4}$$

Inverse Laplace gives transient response or time solution,

$$y(t) = 1 + 2e^{-3t} - 3e^{-4t}$$

Using Initial -Value Theorem,

$$y(0) = \lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} s\frac{6(s+2)}{s(s^2 + 7s + 12)} = 0$$

Using Final -Value Theorem,

$$y(\infty) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\frac{6(s+2)}{s(s^2 + 7s + 12)} = 1$$

<u>Verification,</u>

$$y(t) = 1 + 2e^{-3t} - 3e^{-4t}$$

Value of transient response at zero time,

$$y(0) = 1 + 2e^{-3(0)} - 3e^{-4(0)} = 0$$

Value of steady-state response as time approaches infinity,

$$y(\infty) = 1 + 2e^{-3(\infty)} - 3e^{-4(\infty)} = 1$$

Example 3:

Determine the time solution of the following differential equation when all initial conditions are zero and forcing function is a ramp function as $f(t) = 4t$,

$$y(t) = \frac{1}{D^2 + 3D + 2} f(t)$$

Also using Initial and Final value Theorems to determine value of transient response at zero time $y(0)$ and value of steady-state response $y(\infty)$

Solution:

Since all the initial conditions are zero,

$$Y(s) = \frac{1}{s^2 + 3s + 2} F(s) \qquad Y(s) = \frac{1}{(s+1)(s+2)} F(S)$$

$$f(t) = 4t \qquad F(s) = \mathcal{L}[f(t)] = \mathcal{L}[4t] = 4\mathcal{L}[t] = \frac{4}{s^2}$$

Laplace transformed equation,

$$Y(s) = \frac{4}{s^2(s+1)(s+2)}$$

Using the techniques of Partial Fraction Expansion:

$$Y(s) = \frac{C_q}{(s-r)^q} + \frac{C_{q-1}}{(s-r)^{q-1}} + \cdots + \frac{C_1}{s-r} + \frac{K_1}{s-r_1} + \cdots + \frac{K_{n-q}}{s-r_{n-q}}$$

$$Y(s) = \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$C_q = \lim_{s\to r}[(s-r)^q Y(s)]$$

$$C_2 = \lim_{s\to 0}\left[s^2 \frac{4}{s^2(s+1)(s+2)}\right] = 2$$

$$C_{q-1} = \lim_{s\to r}\left\{\frac{1}{1!}\frac{d}{dt}[(s-r)^q Y(s)]\right\}$$

$$C_1 = \lim_{s\to 0}\left\{\frac{d}{ds}\left[s^2 \frac{4}{s^2(s+1)(s+2)}\right]\right\} = \lim_{s\to 0}\left\{\frac{d}{ds}\left[\frac{4}{(s+1)(s+2)}\right]\right\}$$

$$C_1 = \lim_{s\to 0}\left\{\frac{d}{ds}\left[\frac{4}{s^2 + 3s + 2}\right]\right\} = \lim_{s\to 0}\left[\frac{-4(2s+3)}{(s^2+3s+2)^2}\right] = -3$$

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{4}{s^2(s+1)(s+2)}\right] = 4$$

$$K_2 = \lim_{s\to -2}\left[(s+2)\frac{4}{s^2(s+1)(s+2)}\right] = -1$$

$$Y(s) = \frac{2}{s^2} - \frac{3}{s} + \frac{4}{s+1} - \frac{1}{s+2}$$

Inverse Laplace gives transient response or time solution,

$$y(t) = 2t - 3 + 4e^{-t} - e^{-2t}$$

An alternative technique to evaluate C_1 in order to avoid differentiation,

$$Y(s) = \frac{4}{s^2(s+1)(s+2)} = \frac{2}{s^2} + \frac{C_1}{s} + \frac{4}{s+1} - \frac{1}{s+2}$$

This equation is valid for any value of s. For $s = 1$,

$$Y(1) = \frac{4}{(1)(2)(3)} = \frac{2}{1} + \frac{C_1}{1} + \frac{4}{2} - \frac{1}{3} \qquad C_1 = -3$$

Using Initial -Value Theorem,

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{4}{s^2(s+1)(s+2)} = 0$$

Using Final -Value Theorem,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{4}{s^2(s+1)(s+2)} = \infty$$

Verification,

$$y(t) = 2t - 3 + 4e^{-t} - e^{-2t}$$

Value of transient response at zero time,

$$y(0) = 2(0) - 3 + 4e^{-(0)} - e^{-2(0)} = 0$$

Value of steady-state response as time approaches infinity,

$$y(\infty) = 2(\infty) - 3 + 4e^{-(\infty)} - e^{-2(\infty)} = \infty$$

Example 4:

Determine the time solution of the following differential equation when all initial conditions are zero and forcing function is a ramp function as $f(t) = 4t$,

$$y(t) = \frac{10(D+2)}{D^2 + 9D + 20} f(t)$$

Also using Initial and Final value Theorems to determine value of transient response at zero time $y(0)$ and value of steady-state response $y(\infty)$

Solution:

Since all the initial conditions are zero:

$$Y(s) = \frac{10(s+2)}{s^2 + 9s + 20} F(s)$$

$s^2 + 9s + 20 = (s+4)(s+5) = 0$ Characteristic equation

$$Y(s) = \frac{10(s+2)}{(s+4)(s+5)} F(S)$$

Since, $f(t) = 4t$, $F(s) = \mathcal{L}[f(t)] = \mathcal{L}[4t] = 4\mathcal{L}[t] = \frac{4}{s^2}$

Laplace transformed equation,

$$Y(s) = \frac{40(s+2)}{s^2(s+4)(s+5)}$$

Techniques of Partial-Fraction-Expansion,

$$Y(s) = \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{K_1}{s+4} + \frac{K_2}{s+5}$$

$$C_2 = \lim_{s\to 0}\left[s^2 \frac{40(s+2)}{s^2(s+4)(s+5)}\right] = 4$$

$$C_1 = \lim_{s\to 0}\left\{\frac{d}{ds}\left[s^2 \frac{40(s+2)}{s^2(s+4)(s+5)}\right]\right\} = \lim_{s\to 0}\left\{\frac{d}{ds}\left[\frac{40(s+2)}{(s+4)(s+5)}\right]\right\} = \frac{1}{5}$$

$$K_1 = \lim_{s\to -4}\left[(s+4)\frac{40(s+2)}{s^2(s+4)(s+5)}\right] = -5$$

$$K_2 = \lim_{s\to -5}\left[(s+5)\frac{40(s+2)}{s^2(s+4)(s+5)}\right] = \frac{24}{5}$$

$$Y(s) = \frac{4}{s^2} + \frac{1/5}{s} - \frac{5}{s+4} + \frac{24/5}{s+5}$$

Inverse Laplace gives transient response or time solution,

$$y(t) = 4t + \frac{1}{5} - 5e^{-4t} + \frac{24}{5}e^{-5t}$$

An alternative technique to evaluate C_1 in order to avoid differentiation,

$$Y(s) = \frac{40(s+2)}{s^2(s+4)(s+5)} = \frac{4}{s^2} + \frac{C_1}{s} - \frac{5}{s+4} + \frac{24/5}{s+5}$$

This equation is valid for any value of s. For $s = 1$,

$$Y(1) = \frac{120}{(1)(5)(6)} = \frac{4}{1} + \frac{C_1}{1} - \frac{5}{5} + \frac{24/5}{6} \qquad C_1 = \frac{1}{5}$$

Using Initial -Value Theorem,

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{40(s+2)}{s^2(s+4)(s+5)} = 0$$

Using Final -Value Theorem,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{40(s+2)}{s^2(s+4)(s+5)} = \infty$$

Verification,

$$y(t) = 4t + \frac{1}{5} - 5e^{-4t} + \frac{24}{5}e^{-5t}$$

Value of transient response at zero time,

$$y(0) = 4(0) + \frac{1}{5} - 5e^{-4(0)} + \frac{24}{5}e^{-5(0)} = 0$$

Value of steady-state response as time approaches infinity,

$$y(\infty) = 4(\infty) + \frac{1}{5} - 5e^{-4(\infty)} + \frac{24}{5}e^{-5(\infty)} = \infty$$

Example 5: Determine the time solution of the following differential equation when all initial conditions are zero and $f(t) = e^{-t}$:

$$y(t) = \frac{1}{(D+1)^2(D+2)} f(t)$$

Also using Initial and Final value Theorems to determine value of transient response at zero time $y(0)$ and value of steady-state response $y(\infty)$

Solution: Since all initial conditions are zero, Laplace Transform yields,

$$Y(s) = \frac{1}{(s+1)^2(s+2)} F(s)$$

$$f(t) = e^{-t} \qquad F(s) = \mathcal{L}[f(t)] = \mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

Laplace transformed equation,

$$Y(s) = \frac{1}{(s+1)^3(s+2)}$$

Techniques of Partial-Fraction-Expansion,

$$Y(s) = \frac{C_3}{(s+1)^3} + \frac{C_2}{(s+1)^2} + \frac{C_1}{s+1} + \frac{K_1}{s+2}$$

$$C_3 = \lim_{s\to-1}\left[(s+1)^3 \frac{1}{(s+1)^3(s+2)}\right] = 1$$

$$C_2 = \lim_{s\to-1}\left\{\frac{d}{ds}\left[(s+1)^3 \frac{1}{(s+1)^3(s+2)}\right]\right\} = \lim_{s\to-1}\left\{\left[\frac{-1}{(s+2)^2}\right]\right\} = -1$$

$$C_1 = \lim_{s\to-1}\left\{\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3 \frac{1}{(s+1)^3(s+2)}\right]\right\} = \lim_{s\to-1}\left\{\frac{1}{2}\left[\frac{2}{(s+2)^3}\right]\right\} = 1$$

$$K_1 = \lim_{s\to-2}\left[(s+2) \frac{1}{(s+1)^3(s+2)}\right] = -1$$

$$Y(s) = \frac{1}{(s+1)^3} - \frac{1}{(s+1)^2} + \frac{1}{s+1} - \frac{1}{s+2}$$

Inverse Laplace gives transient response or time solution,

$$y(t) = \frac{1}{2}t^2e^{-t} - te^{-t} + e^{-t} - e^{-2t} = \left[1 - t + \frac{1}{2}t^2\right]e^{-t} - e^{-2t}$$

An alternative technique to evaluate C_1 and C_2 to avoid differentiation,

$$Y(s) = \frac{1}{(s+1)^3(s+2)} = \frac{1}{(s+1)^3} + \frac{C_2}{(s+1)^2} + \frac{C_1}{s+1} - \frac{1}{s+2}$$

$$Y(0) = \frac{1}{(1)^3(2)} = \frac{1}{(1)^3} + \frac{C_2}{(1)^2} + \frac{C_1}{1} - \frac{1}{2} \qquad Or, \quad C_1 + C_2 = 0$$

$$Y(1) = \frac{1}{(2)^3(3)} = \frac{1}{(2)^3} + \frac{C_2}{(2)^2} + \frac{C_1}{2} - \frac{1}{3} \qquad Or, \quad 2\,C_1 + C_2 = 1$$

Solve equations simultaneously, $C_1 = 1,$ and $C_2 = -1$

Using Initial -Value Theorem to determine the value of transient response at zero time,

$$y(0) = \lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{1}{(s+1)^3(s+2)} = 0$$

Using Final -Value Theorem to determine the value of the steady-state response,

$$y(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{1}{(s+1)^3(s+2)} = \infty$$

Verification,

$$y(t) = \left[1 - t + \frac{1}{2}t^2\right]e^{-t} - e^{-2t}$$

Value of transient response at zero time,

$$y(0) = \left[1 - (0) + \frac{1}{2}(0)^2\right]e^{-(0)} - e^{-2(0)} = 0$$

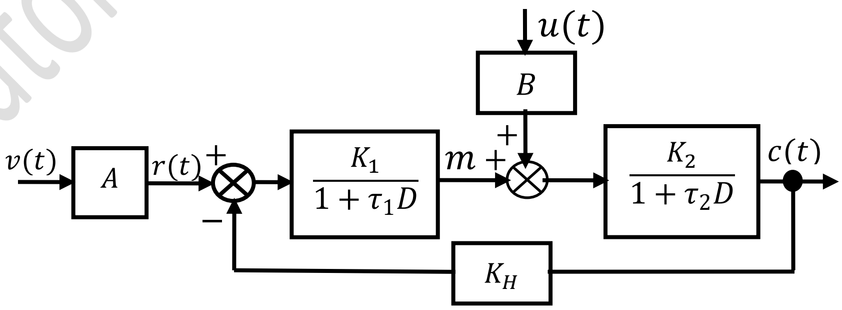
Value of steady-state response as time approaches infinity,

$$y(\infty) = \left[1 - (\infty) + \frac{1}{2}(\infty)^2\right]e^{-(\infty)} - e^{-2(\infty)} = 0$$

Example 6:

In block-diagram representation for a proportional control system shown in **Figure**, $A = 1,\ K_1 = 1,\ K_2 = 2,\ K_H = 0.5,\ \ B = -5,\ \tau_1 = \frac{1}{6}, and\ \tau_2 = 1.$ Determine the transient response and its values at zero time $c(0)$ and at steady-state response $c(\infty)$ by using Initial and Final value Theorems when all initial conditions are zero for each of the following cases,

a) v is a step function of constant value v, and $u = 0$.
b) u is a step function of constant value u, and $v = 0$.



Solution:

General differential equation of operation for a control system with two inputs and one output is:

$$c(t) = \frac{N_{G1}N_{G2}D_H}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}r(t) + \frac{N_{G2}D_H D_{G1}}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}d(t)$$

Then,

$$c(t)=\frac{AK_1K_2}{K_1K_2K_H+(1+\tau_1D)(1+\tau_2D)}v(t)+\frac{BK_2(1+\tau_1D)}{K_1K_2K_H+(1+\tau_1D)(1+\tau_2D)}u(t)$$

Substitute numerical values,

$$c(t)=\frac{2}{1+\left(1+\frac{1}{6}D\right)(1+D)}v(t)-\frac{10\left(1+\frac{1}{6}D\right)}{1+\left(1+\frac{1}{6}D\right)(1+D)}u(t)$$

$$c(t)=\frac{12}{D^2+7D+12}v(t)-\frac{10(6+D)}{D^2+7D+12}u(t)$$

$$c(t)=\frac{12}{(D+3)(D+4)}v(t)-\frac{10(6+s)}{(D+3)(D+4)}u(t)$$

Since all initial conditions are zero, Laplace transforms could be obtained by replacing D by s:

$$C(s)=\frac{12}{(s+3)(s+4)}V(s)-\frac{10(6+s)}{(s+3)(s+4)}U(s)$$

a) Since v is a step function of constant value v,and $u=0$, then $V(s)=v/s$ and $U(s)=0$

$$C(s)=\frac{12}{s(s+3)(s+4)}v$$

Techniques of Partial-Fraction-Expansion,

$$C(s)=\left[\frac{K_1}{s}+\frac{K_2}{s+3}+\frac{K_3}{s+4}\right]v$$

$$K_i=\lim_{s\to r_i}[(s-r_i)C(s)]$$

$$K_1=\lim_{s\to 0}\left[(s-0)\frac{12}{s(s+3)(s+4)}\right]=1$$

$$K_2=\lim_{s\to -3}\left[(s+3)\frac{12}{s(s+3)(s+4)}\right]=-4$$

$$K_3=\lim_{s\to -4}\left[(s+4)\frac{12}{s(s+3)(s+4)}\right]=3$$

$$C(s)=\left[\frac{1}{s}-\frac{4}{s+3}+\frac{3}{s+4}\right]v$$

Inverse Laplace gives transient response or time solution,

$$c(t)=(1-4e^{-3t}+3e^{-4t})v$$

Using Initial -Value Theorem,

$$c(0)=\lim_{t\to 0}c(t)=\lim_{s\to\infty}sC(s)=\lim_{s\to\infty}s\frac{12}{s(s+3)(s+4)}v=0$$

Using Final -Value Theorem,

$$c(\infty)=\lim_{t\to\infty}c(t)=\lim_{s\to 0}sC(s)=\lim_{s\to 0}s\frac{12}{s(s+3)(s+4)}v=v$$

Verification,

$$c(t) = (1 - 4e^{-3t} + 3e^{-4t})v$$

Value of transient response at zero time,

$$c(0) = \left(1 - 4e^{-3(0)} + 3e^{-4(0)}\right)v = 0$$

Value of steady-state response as time approaches infinity,

$$c(\infty) = \left(1 - 4e^{-3(\infty)} + 3e^{-4(\infty)}\right)v = v$$

b) Since u is a step function of constant value u, and $v = 0$, then $U(s) = u/s$ and $V(s) = 0$

$$C(s) = -\frac{10(6+s)}{s(s+3)(s+4)}u$$

Techniques of Partial-Fraction-Expansion,

$$C(s) = \left[\frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{s+4}\right]u$$

$$K_i = \lim_{s\to r_i}[(s - r_i)C(s)]$$

$$K_1 = \lim_{s\to 0}\left[(s-0)\frac{-10(6+s)}{s(s+3)(s+4)}\right] = -5$$

$$K_2 = \lim_{s\to -3}\left[(s+3)\frac{-10(6+s)}{s(s+3)(s+4)}\right] = 10$$

$$K_3 = \lim_{s\to -4}\left[(s+4)\frac{-10(6+s)}{s(s+3)(s+4)}\right] = -5$$

$$C(s) = \left[-\frac{5}{s} + \frac{10}{s+3} - \frac{5}{s+4}\right]u$$

Inverse Laplace gives transient response or time solution,

$$c(t) = (-5 + 10e^{-3t} - 5e^{-4t})u$$

Using Initial -Value Theorem,

$$c(0) = \lim_{t\to 0} c(t) = \lim_{s\to\infty} sC(s) = \lim_{s\to\infty} s\left[-\frac{10(6+s)}{s(s+3)(s+4)}u\right] = 0$$

Using Final -Value Theorem,

$$c(\infty) = \lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\left[-\frac{10(6+s)}{s(s+3)(s+4)}u\right] = -5u$$

Verification,

$$c(t) = (-5 + 10e^{-3t} - 5e^{-4t})u$$

Value of transient response at zero time,

$$c(0) = \left(-5 + 10e^{-3(0)} - 5e^{-4(0)}\right)u = 0$$

Value of steady-state response as time approaches infinity,

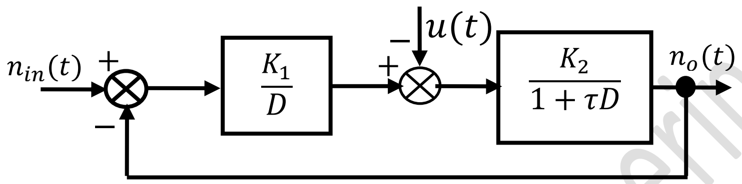
$$c(\infty) = \left(-5 + 10e^{-3(\infty)} - 5e^{-4(\infty)}\right)u = -5u$$

Example 7:

Block-diagram representation for an integral speed control system shown in Figure. The output speed is n_o, the desired input speed is n_{in}, and the load torque is u. For $\tau = 0.25$, $K_1 = 1$, $K_2 = 0.75$, and all initial conditions are zero, determine the transient response of the control system for each of the following cases:

a) n_{in} is a step function of constant value n_{in}, and $u = 0$.

b) u is a step function of constant value u, and $n_{in} = 0$.



Solution:

General differential equation of operation for a control system with two inputs and one output is:

$$c(t) = \frac{N_{G1}N_{G2}D_H}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H} r(t) + \frac{N_{G2}D_H D_{G1}}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H} d(t)$$

Then,

$$n_o(t) = \frac{K_1K_2}{K_1K_2 + D(1+\tau D)} n_{in}(t) - \frac{K_2 D}{K_1K_2 + D(1+\tau D)} u(t)$$

Substitute numerical values yields;

$$n_o(t) = \frac{0.75}{0.75 + D(1+0.25D)} n_{in}(t) - \frac{0.75D}{0.75 + D(1+0.25D)} u(t)$$

$$n_o(t) = \frac{3}{D^2 + 4D + 3} n_{in}(t) - \frac{3D}{D^2 + 4D + 3} u(t)$$

$$n_o(t) = \frac{3}{(D+1)(D+3)} n_{in}(t) - \frac{3D}{(D+1)(D+3)} u(t)$$

Since all initial conditions are zero, Laplace transforms could be obtained by replacing D by s:

$$N_o(s) = \frac{3}{(s+1)(s+3)} N_{in}(s) - \frac{3s}{(s+1)(s+3)} U(s)$$

a) Since n_{in} is a step function of constant value n_{in}, and $u = 0$, then

$N_{in}(s) = n_{in}/s$ and $U(s) = 0$

$$N_o(s) = \frac{3}{s(s+1)(s+3)} n_{in}$$

Techniques of Partial-Fraction-Expansion:

$$N_o(s) = \left[\frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3}\right] n_{in}$$

$K_i = \lim_{s \to r_i} [(s - r_i) N_o(s)]$

$K_1 = \lim_{s \to 0} \left[s \frac{3}{s(s+1)(s+3)} \right] = 1$

$K_2 = \lim_{s \to -1} \left[(s+1) \frac{3}{s(s+1)(s+3)} \right] = -\frac{3}{2}$

$K_3 = \lim_{s \to -3} \left[(s+3) \frac{3}{s(s+1)(s+3)} \right] = \frac{1}{2}$

$$N_o(s) = \left[\frac{1}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3} \right] n_{in}$$

Inverse Laplace gives transient response or time solution,

$$n_o(t) = \left(1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t} \right) n_{in}$$

Using Initial -Value Theorem,

$$n_o(0) = \lim_{t \to 0} n_o(t) = \lim_{s \to \infty} s N_o(s) = \lim_{s \to \infty} s \frac{3}{s(s+1)(s+3)} n_{in} = 0$$

Using Final -Value Theorem,

$$n_o(\infty) = \lim_{t \to \infty} n_o(t) = \lim_{s \to 0} s N_o(s) = \lim_{s \to 0} s \frac{3}{s(s+1)(s+3)} n_{in} = n_{in}$$

<u>Verification,</u>

$$n_o(t) = \left(1 - \frac{3}{2} e^{-t} + \frac{1}{2} e^{-3t} \right) n_{in}$$

Value of transient response at zero time,

$$n_o(0) = \left(1 - \frac{3}{2} e^{-(0)} + \frac{1}{2} e^{-3(0)} \right) n_{in} = 0$$

Value of steady-state response as time approaches infinity,

$$n_o(\infty) = \left(1 - \frac{3}{2} e^{-(\infty)} + \frac{1}{2} e^{-3(\infty)} \right) n_{in} = n_{in}$$

b) Since u is a step function of constant value u, and $n_{in} = 0$, then
$U(s) = u/s$ and $N_{in}(s) = 0$

$$N_o(s) = \frac{-3}{(s+1)(s+3)} u$$

Using techniques of Partial-Fraction-Expansion:

$$N_o(s) = \left[\frac{K_1}{s+1} + \frac{K_2}{s+3} \right] u$$

$K_1 = \lim_{s \to -1} \left[(s+1) \frac{-3}{(s+1)(s+3)} \right] = -\frac{3}{2}$

$K_2 = \lim_{s \to -3} \left[(s+3) \frac{-3}{(s+1)(s+3)} \right] = \frac{3}{2}$

$$N_o(s) = \left[-\frac{3/2}{s+1} + \frac{3/2}{s+3} \right] u$$

Inverse Laplace gives transient response or time solution,

$$n_o(t) = \left(-\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t}\right)u$$

Using Initial -Value Theorem,

$$n_o(0) = \lim_{t\to 0} n_o(t) = \lim_{s\to\infty} sN_o(s) = \lim_{s\to\infty} s\frac{-3}{(s+1)(s+3)}u = 0$$

Using Final -Value Theorem,

$$n_o(\infty) = \lim_{t\to\infty} n_o(t) = \lim_{s\to 0} sN_o(s) = \lim_{s\to 0} s\frac{-3}{(s+1)(s+3)}u = 0$$

<u>Verification,</u>

$$n_o(t) = \left(-\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t}\right)u$$

Value of transient response at zero time,

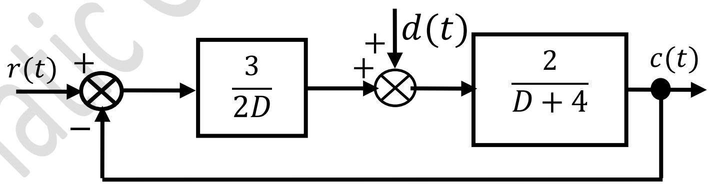
$$n_o(0) = \left(-\frac{3}{2}e^{-(0)} + \frac{3}{2}e^{-3(0)}\right)u = 0$$

Value of steady-state response as time approaches infinity,

$$n_o(\infty) = \left(-\frac{3}{2}e^{-(\infty)} + \frac{3}{2}e^{-3(\infty)}\right)u = 0$$

<u>Example 8:</u>

The system shown in **Figure** is initially at equilibrium, with $r = 1$ and $d = 0$. A step-function disturbance $d(t) = u(t)$ is then initiated at time $t = 0$. Determine the response $c(t)$ for $t > 0$.



<u>Solution:</u>

<u>Q1:</u>

For the system shown in Figure, determine the response $c(t)$ when,

a) $r(t) = u(t)$, $d(t) = 0$ and $c(0) = c'(0) = 0$
b) $r(t) = 0$, $d(t) = u(t)$ and $c(0) = 1$ and $c'(0) = -1$

General differential equation of operation for a control system with two inputs and one output,

$$c(t) = \frac{N_{G1}N_{G2}D_H}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}r(t) + \frac{N_{G2}D_H D_{G1}}{N_{G1}N_{G2}N_H + D_{G1}D_{G2}D_H}d(t)$$

$$c(t) = \frac{3(2)}{(3)(2) + 2D(D+4)}r(t) + \frac{2(2D)}{(3)(2) + 2D(D+4)}d(t)$$

$$c(t) = \frac{3}{D^2+4D+3} r(t) + \frac{2D}{D^2+4D+3} d(t)$$

$$c(t) = \frac{3r(t)+2Dd(t)}{D^2+4D+3}$$

$D^2c(t) + 4Dc(t) + 3c(t) = 3r(t) + 2Dd(t)$

Since the system is initially at equilibrium means that at $t = 0$, $D = 0$,

$$(0)^2c(0) + 4(0)c(0) + 3c(0) = 3r(0) + 2(0)d(t)$$

$3c(0) = 3r(0)$, $\quad c(0) = r(0) = 1 \quad c'(0) = 0 \quad d(0) = 0$

Laplace transforms,

$[s^2C(s) - sc(0) - c'(0)] + 4[sC(s) - c(0)] + 3C(s) = 3R(s) + 2[sD(s) - d(0)]$

$$s^2C(s) - s + 4sC(s) - 4 + 3C(s) = 3R(s) + 2sD(s)$$

$$(s^2+4s+3)C(s) = 3R(s) + 2sD(s) + s + 4$$

Since $r(t) = 1$, then $R(s) = \mathcal{L}[1] = \frac{1}{s}$

Also $d(t) = u(t)$ is a step function disturbance, $D(s) = \mathcal{L}[u(t)] = \frac{1}{s}$

$$(s^2+4s+3)C(s) = \frac{3}{s} + 2 + s + 6$$

$$C(s) = \frac{s^2+8s+3}{s(s^2+4s+3)} = \frac{s^2+8s+3}{s(s+1)(s+3)}$$

Techniques of Partial-Fraction-Expansion,

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3}$$

$K_1 = \lim_{s\to 0}\left[s\frac{s^2+8s+3}{s(s+1)(s+3)}\right] = 1$

$K_2 = \lim_{s\to -1}\left[(s+1)\frac{s^2+8s+3}{s(s+1)(s+3)}\right] = 2$

$K_3 = \lim_{s\to -3}\left[(s+3)\frac{s^2+8s+3}{s(s+1)(s+3)}\right] = -2$

$$C(s) = \frac{1}{s} + \frac{2}{s+1} - \frac{2}{s+3}$$

Inverse Laplace gives transient response or time solution,

$$c(t) = 1 + 2e^{-t} - 2e^{-3t}$$

Using Initial -Value Theorem,

$$c(0) = \lim_{t\to 0} c(t) = \lim_{s\to\infty} sC(s) = \lim_{s\to\infty} s\frac{s^2+8s+3}{s(s^2+4s+3)} = 1$$

Using Final -Value Theorem,

$$c(\infty) = \lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\frac{s^2+8s+3}{s(s^2+4s+3)} = 1$$

<u>Verification,</u>

$$c(t) = 1 + 2e^{-t} - 2e^{-3t}$$

Value of transient response at zero time,

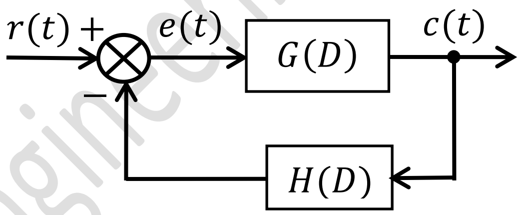
$$c(0) = 1 + 2e^{-(0)} - 2e^{-3(0)} = 1$$

Value of steady-state response as time approaches infinity,

$$c(\infty) = 1 + 2e^{-(\infty)} - 2e^{-3(\infty)} = 1$$

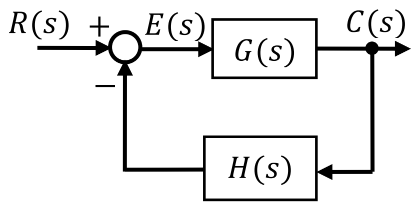
Performance of Control System:

Steady state error is used to measure the accuracy of a control system. It is the difference between the reference input and actual output as time approaches infinity. Consider the simple (SISO) control system shown in **Figure**,



$$e(t) = r(t) - H(D)c(t)$$
$$e(t) = r(t) - G(D)H(D)e(t)$$
$$e(t) = \frac{r(t)}{1 + G(D)H(D)}$$

For all initial conditions are zero, Laplace transformed equation as shown in **Figure**,



$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

Using Final-Value Theorem,

$$e_{ss} = e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1 + G(s)H(s)}$$

1. Unit step input $[r(t) = u(t), \quad R(s) = 1/s]$

$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1 + G(s)H(s)} = \lim_{s\to 0} s\frac{1/s}{1 + G(s)H(s)} = \frac{1}{1 + \lim\limits_{s\to 0} G(s)H(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}$$

$K_p = \lim\limits_{s\to 0} G(s)H(s)$ Positional error constant

2. Unit ramp input $[r(t) = t, \quad R(s) = 1/s^2]$

$$e_{ss} = \lim_{s\to 0} s\frac{1/s^2}{1 + G(s)H(s)} = \lim_{s\to 0}\left[\frac{1}{s + sG(s)H(s)}\right] = \frac{1}{\lim\limits_{s\to 0} sG(s)H(s)} = \frac{1}{K_v}$$

$$e_{ss} = \frac{1}{K_v}$$

$K_v = \lim\limits_{s\to 0} sG(s)H(s)$ Velocity error constant

1. Unit parabolic input $[r(t) = t^2/2, \quad R(s) = 1/s^3]$

$$e_{ss} = \lim_{s\to 0} s\frac{1/s^3}{1+G(s)H(s)} = \lim_{s\to 0}\left[\frac{1}{s^2+s^2G(s)H(s)}\right] = \frac{1}{\lim_{s\to 0} s^2G(s)H(s)} = \frac{1}{K_a}$$

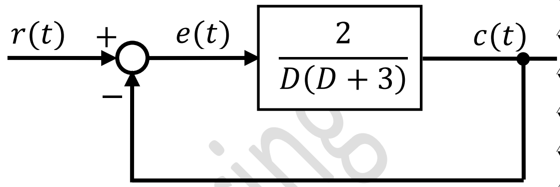
$$e_{ss} = \frac{1}{K_a}$$

$K_a = \lim_{s\to 0} s^2G(s)H(s)$ Acceleration error constant

Where, K_p, K_v, and K_a, are called error coefficients

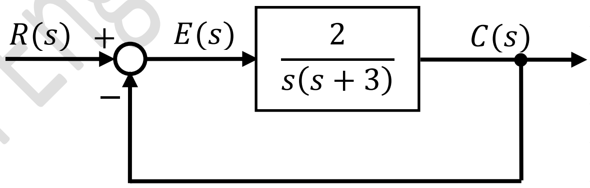
Example 9:

For control system shown in **Figure**, determine the response to a unit step function, a unit ramp function, and a unit parabolic function when all initial conditions are zero. What is the steady-state error to each of these inputs?



Solution:

Since all initial conditions are zero, Laplace block diagram representation as shown in **Figure**,



$$\frac{C(s)}{R(s)} = \frac{\frac{2}{s(s+3)}}{1+\frac{2}{s(s+3)}} = \frac{2}{s^2+3s+2}$$

$$C(s) = \frac{2}{s^2+3s+2}R(s)$$

$$C(s) = \frac{2}{(s+1)(s+2)}R(s)$$

1. A unit step input, $r(t) = u(t)$ and $R(s) = 1/s$

$$C(s) = \frac{2}{s(s+1)(s+2)}$$

Partial fraction expansion,

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

$$K_1 = \lim_{s\to 0}\left[s\frac{2}{s(s+1)(s+2)}\right] = 1$$

$$K_2 = \lim_{s\to -1}\left[(s+1)\frac{2}{s(s+1)(s+2)}\right] = -2$$

$$K_3 = \lim_{s\to -2}\left[(s+2)\frac{2}{s(s+1)(s+2)}\right] = 1$$

$$C(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

Laplace inverse gives response to a unit step input,

$$c(t) = 1 - 2e^{-t} + e^{-2t}$$

Steady state error for a unit step input is determined as:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{2}{s(s+3)} = \infty \qquad e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

Also steady-state error e_{ss} could be obtained as,

$$e(t) = r(t) - c(t) = 1 - (1 - 2e^{-t} + e^{-2t}) = 2e^{-t} - e^{-2t}$$

Steady-state error,

$$e_{ss} = e(\infty) = 2e^{-(\infty)} - e^{-2(\infty)} = 0$$

2. A unit ramp input, $r(t) = t$ and $R(s) = 1/s^2$

$$C(s) = \frac{2}{s^2(s+1)(s+2)}$$

Partial fraction expansion,

$$C(s) = \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$C_q = \lim_{s\to r}[(s-r)^q C(s)]$$

$$C_2 = \lim_{s\to 0}\left[s^2 \frac{2}{s^2(s+1)(s+2)}\right] = 1$$

$$K_i = \lim_{s\to r_i}[(s-r_i)C(s)]$$

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{2}{s^2(s+1)(s+2)}\right] = 2$$

$$K_2 = \lim_{s\to -2}\left[(s+2)\frac{2}{s^2(s+1)(s+2)}\right] = -\frac{1}{2}$$

$$C(s) = \frac{2}{s^2(s+1)(s+2)} = \frac{1}{s^2} + \frac{C_1}{s} + \frac{2}{s+1} - \frac{1/2}{s+2}$$

For, $s = 1$

$$C(1) = \frac{2}{(1)(2)(3)} = \frac{1}{1} + \frac{C_1}{1} + \frac{2}{2} - \frac{1/2}{3} \qquad \text{and,} \qquad C_1 = -\frac{3}{2}$$

$$C(s) = \frac{1}{s^2} - \frac{3/2}{s} + \frac{2}{s+1} - \frac{1/2}{s+2}$$

Laplace inverse gives response to a unit ramp input,

$$c(t) = t - \frac{3}{2} + 2e^{-t} - \frac{1}{2}e^{-2t}$$

Steady state error for a unit ramp input is determined as:

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} s\frac{2}{s(s+3)} = \frac{2}{3}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{2/3} = \frac{3}{2}$$

Also steady-state error e_{ss} could be obtained as,

$$e(t) = r(t) - c(t) = t - \left(t - \frac{3}{2} + 2e^{-t} - \frac{1}{2}e^{-2t}\right) = \frac{3}{2} - 2e^{-t} + \frac{1}{2}e^{-2t}$$

Steady-state error,

$$e_{ss} = e(\infty) = \frac{3}{2} - 2e^{-(\infty)} + \frac{1}{2}e^{-2(\infty)} = \frac{3}{2}$$

3. A unit parabolic input, $r(t) = t^2/2$ and $R(s) = 1/s^3$

$$C(s) = \frac{2}{s^3(s+1)(s+2)}$$

Partial fraction expansion,

$$C(s) = \frac{C_3}{s^3} + \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$C_q = \lim_{s\to r}[(s-r)^q C(s)]$$

$$C_3 = \lim_{s\to 0}\left[(s+0)^3 \frac{2}{s^3(s+1)(s+2)}\right] = 1$$

$$K_i = \lim_{s\to r_i}[(s-r_i)C(s)]$$

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{2}{s^3(s+1)(s+2)}\right] = -2$$

$$K_2 = \lim_{s\to -2}\left[(s+2)\frac{2}{s^3(s+1)(s+2)}\right] = \frac{1}{4}$$

$$C(s) = \frac{2}{s^3(s+1)(s+2)} = \frac{1}{s^3} + \frac{C_2}{s^2} + \frac{C_1}{s} - \frac{2}{s+1} + \frac{1/4}{s+2}$$

For $s = 1$ and $s = 2$

$$C(1) = \frac{2}{(1)^3(2)(3)} = \frac{1}{(1)^3} + \frac{C_2}{(1)^2} + \frac{C_1}{1} - \frac{2}{2} + \frac{1/4}{3} \qquad \text{Or,} \quad C_1 + C_2 = \frac{1}{4}$$

$$C(2) = \frac{2}{(2)^3(3)(4)} = \frac{1}{(2)^3} + \frac{C_2}{(2)^2} + \frac{C_1}{2} - \frac{2}{3} + \frac{1/4}{4} \qquad \text{Or,} \quad 2C_1 + C_2 = 2$$

Solve equations simultaneously,

$$C_1 = \frac{7}{4}, \quad \text{and} \quad C_2 = -\frac{3}{2}$$

$$C(s) = \frac{1}{s^3} - \frac{3/2}{s^2} + \frac{7/4}{s} - \frac{2}{s+1} + \frac{1/4}{s+2}$$

Laplace inverse gives response to a unit parabolic input,

$$c(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4} - 2e^{-t} + \frac{1}{4}e^{-2t}$$

Steady state error for a unit parabolic input is determined as:

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} s^2 \frac{2}{s(s+3)} = \lim_{s\to 0}\frac{2}{1+3/s} = \frac{2}{1+\infty} = 0$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Also steady-state error e_{ss} could be obtained as,

$$e(t) = r(t) - c(t) = \frac{1}{2}t^2 - \left(\frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4} - 2e^{-t} + \frac{1}{4}e^{-2t}\right)$$

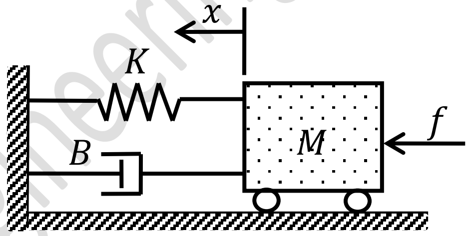
$$e(t) = \frac{3}{2}t - \frac{7}{4} + 2e^{-t} - \frac{1}{4}e^{-2t}$$

Steady-state error,

$$e_{ss} = e(\infty) = \frac{3}{2}(\infty) - \frac{7}{4} + 2e^{-(\infty)} - \frac{1}{4}e^{-2(\infty)} = \infty$$

<u>Example 10:</u>

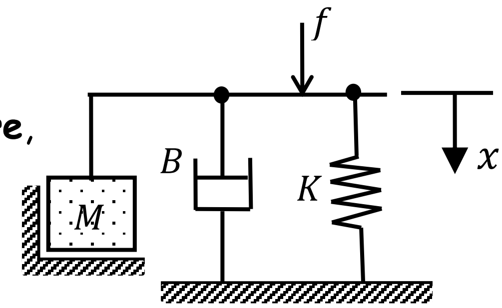
A mass-spring damper system is shown in **Figure**:



a. Write the mathematical differential equation of operation.
b. Using Laplace transform to solve the equation when $x(0) = 0$, $x'(0) = 1$, $f = 0$, $M = 1$, $B = 3$, and $K = 2$.
c. Determine the steady state error when $(f = 0)$
d. Determine the steady state error for a unit step input, unit ramp input and unit parabolic input.

<u>Solution:</u>

A grounded chair representation as shown in **Figure**,



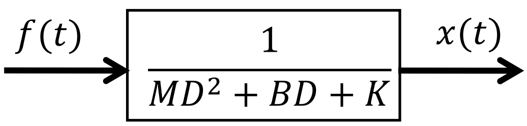
a) Mathematical differential equation,

$Z = Z_1 + Z_2 + Z_3$ $\qquad Z = MD^2 + BD + K$

$f = Zx$

$f = (MD^2 + BD + K)x$

$$x(t) = \frac{1}{MD^2+BD+K}f(t)$$



b) Laplace transforms,

$MD^2x(t) + BDx(t) + Kx(t) = f(t)$

$M\left[s^2X(s) - sx(0) - x'(0)\right] + B[sX(s) - x(0)] + KX(s) = F(s)$

$Ms^2X(s) - M + BsX(s) + KX(s) = F(s)$

$$X(s) = \frac{M + F(s)}{MS^2 + BS + K} = \frac{1 + F(s)}{S^2 + 3S + 2} \qquad X(s) = \frac{1 + F(s)}{(s+1)(s+2)}$$

Since, $\quad f = 0, \quad F(s) = \mathcal{L}[0] = 0,$

$$X(s) = \frac{1}{(s+1)(s+2)}$$

Partial-Fraction-Expansion Technique:

$$X(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

$$K_i = \lim_{s \to r_i} [(s - r_i)X(s)]$$

$$K_1 = \lim_{s \to -1} \left[(s+1)\frac{1}{(s+1)(s+2)}\right] = 1$$

$$K_2 = \lim_{s \to -2} \left[(s+2)\frac{1}{(s+1)(s+2)}\right] = -1$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Laplace inverse yields time solution or transient response,

$$x(t) = e^{-t} - e^{-2t}$$

c) Steady-state error

$$e_{ss} = e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

For open loop control system, $H(s) = 0$

$$e_{ss} = e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\left[\frac{F(s)}{1 + G(s)H(s)}\right] = 0$$

Also steady-state error e_{ss} could be obtained as,

$e(t) = f(t)$ Open loop control system

Steady-state error,

$$e_{ss} = e(\infty) = f(\infty) = 0$$

d)

1. Unit step input $[f(t) = u(t) = 1]$

   For open loop control system, $H(s) = 0$

$$K_p = \lim_{s \to 0} G(s)H(s) = 0 \qquad e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + 0} = 1$$

Also steady-state error e_{ss} could be obtained as,

$e(t) = f(t) = u(t) = 1$ Open loop control system

Steady-state error,

$$e_{ss} = e(\infty) = f(\infty) = 1$$

2. Unit ramp input $[f(t) = t]$

   For open loop control system, $H(s) = 0$

$$K_v = \lim_{s \to 0} sG(s)H(s) = 0 \qquad e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty$$

Also steady-state error e_{ss} could be obtained as,

$e(t) = f(t) = t$ Open loop control system

Steady-state error,

$$e_{ss} = e(\infty) = f(\infty) = \infty$$

3. Unit parabolic input $[f(t) = t^2/2]$

   For open loop control system, $H(s) = 0$

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = 0 \qquad e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Also steady-state error e_{ss} could be obtained as,

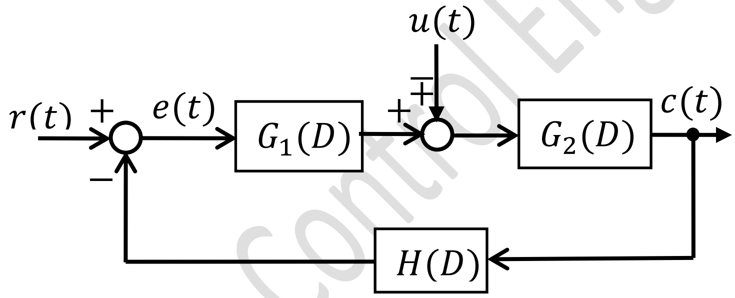
$e(t) = f(t) = \frac{1}{2}t^2$ Open loop control system

Steady-state error,

$$e_{ss} = e(\infty) = f(\infty) = \infty$$

Steady-State Error For Two Inputs-One Output Control System:

Consider a control system with two inputs-one output shown in **Figure**,



$$e(t) = r(t) - H(D)c(t)$$

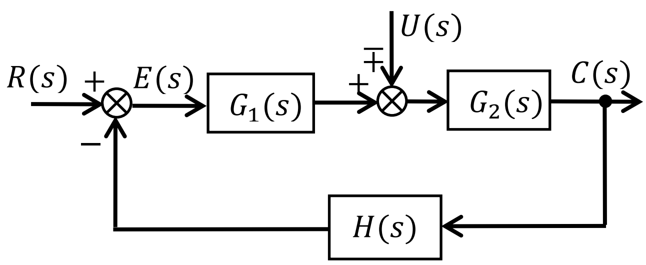
Since,

$$c(t) = \frac{G_1(D)G_2(D)r(t) \mp G_2(D)u(t)}{1 + G_1(D)G_2(D)H(D)}$$

$$e(t) = r(t) - H(D)\left[\frac{G_1(D)G_2(D)r(t) \mp G_2(D)u(t)}{1 + G_1(D)G_2(D)H(D)}\right]$$

$$e(t) = \frac{1}{1 + G_1(D)G_2(D)H(D)}r(t) \pm \frac{G_2(D)H(D)}{1 + G_1(D)G_2(D)H(D)}u(t)$$

For all zero initial conditions, Laplace transformed block diagram shown in Figure and the transformed equation is:



$$E(s)=\frac{1}{1+G_1(s)G_2(s)H(s)}R(s)\pm\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Steady state error using Final-Value Theorem,

$$e_{ss}=\lim_{s\to 0} sE(s)$$

$$e_{ss}=\lim_{s\to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s)\pm\lim_{s\to 0} s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Error coefficients could be determined as:

For unit step input,

$K_p=\lim_{s\to 0} G_1(s)G_2(s)H(s)$ Positional error constant

For unit ramp input,

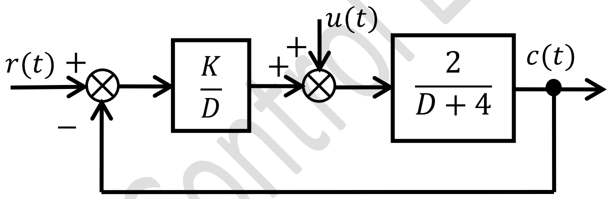
$K_v=\lim_{s\to 0} sG_1(s)G_2(s)H(s)$ Velocity error constant

For unit parabolic input,

$K_a=\lim_{s\to 0} s^2G_1(s)G_2(s)H(s)$ Acceleration error constant

Example 11:

For feedback control system shown in **Figure**, all the initial conditions are zero and $K=1.5$



Determine the transient response and the steady-state error e_{ss} for each of the following:

a. $r(t)$ is a unit step function and $u(t)=0$

b. $u(t)$ is a unit step function and $r(t)=0$

c. $r(t)$ is a unit ramp function $r(t)=t$ and $u(t)=0$

d. Both $r(t)$ and $u(t)$ are unit step functions.

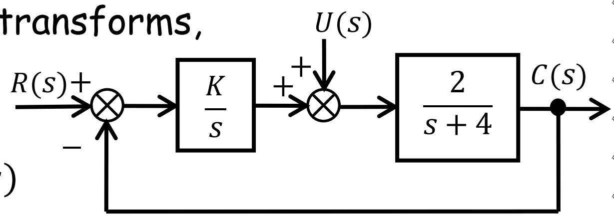
e. $r(t)$ is a unit step function and $u(t)$ is a unit ramp function $u(t)=t$

Solution:

$$C(t)=\frac{N_{G1}N_{G2}D_H r(t)+N_{G2}D_H D_{G1}d(t)}{N_{G1}N_{G2}N_H+D_{G1}D_{G2}D_H}$$

$$C(t)=\frac{(1.5)(2)r(t)+(2)Du(t)}{(1.5)(2)+D(D+4)}\qquad C(t)=\frac{3r(t)+2Du(t)}{D^2+4D+3}$$

Since all initial conditions are zero, Laplace transforms,



$$C(s)=\frac{3R(s)+2sU(s)}{s^2+4s+3}$$

$$C(s)=\frac{3}{(s+1)(s+3)}R(s)+\frac{2s}{(s+1)(s+3)}U(s)$$

a) $r(t)$ is a unit step function and $u(t) = 0$, then $R(s) = 1/s$ and $U(s) = 0$

$$C(s) = \frac{3}{s(s+1)(s+3)}$$

Partial Fraction Expansion,

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3}$$

$$K_1 = \lim_{s\to 0}\left[s\frac{3}{s(s+1)(s+3)}\right] = 1$$

$$K_2 = \lim_{s\to -1}\left[(s+1)\frac{3}{s(s+1)(s+3)}\right] = -\frac{3}{2}$$

$$K_3 = \lim_{s\to -3}\left[(s+3)\frac{3}{s(s+1)(s+3)}\right] = \frac{1}{2}$$

$$C(s) = \frac{1}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}$$

Transient response is obtained by Laplace Inverse,

$$c(t) = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

Steady state error,

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s) - \lim_{s\to 0} s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Since, $R(s) = 1/s$ and $U(s) = 0$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s) = \frac{1}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)} = \frac{1}{1+K_p}$$

$$K_p = \lim_{s\to 0} G_1(s)G_2(s)H(s) = \lim_{s\to 0}\frac{3}{s(s+4)} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

Another way (for a unity feedback only)

$$e(t) = r(t) - c(t)$$

$$e(t) = 1 - \left[1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right] = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

Steady state error,

$$e_{ss} = e(\infty) = -0 + 0 = 0$$

b) $u(t)$ is a unit step function and $r(t) = 0$, then $U(s) = 1/s$

$$C(s) = \frac{2s}{(s+1)(s+3)}U(s) = \frac{2}{(s+1)(s+3)}$$

Partial Fraction Expansion,

$$C(s) = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{2}{(s+1)(s+3)}\right] = 1$$

$$K_2 = \lim_{s\to -3}\left[(s+3)\frac{2}{(s+1)(s+3)}\right] = -1$$

$$C(s) = \frac{1}{s+1} - \frac{1}{s+3}$$

Transient response is obtained by Laplace Inverse,

$$c(t) = e^{-t} - e^{-3t}$$

Steady state error,

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s) - \lim_{s\to 0} s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Since, $R(s) = 0$ and $U(s) = 1/s$

$$e_{ss} = -\lim_{s\to 0}\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}$$

$$e_{ss} = -\lim_{s\to 0}\frac{\frac{2}{(s+4)}}{1+G_1(s)G_2(s)H(s)} = -\frac{\frac{1}{2}}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)}$$

$$K_p = \lim_{s\to 0} G_1(s)G_2(s)H(s) = \lim_{s\to 0}\frac{3}{s(s+4)} = \infty$$

$$e_{ss} = -\frac{1/2}{1+K_p} = -\frac{1/2}{1+\infty} = 0$$

Another way (for a unity feedback only)

$$e(t) = r(t) - c(t)$$

$$e(t) = 0 - [e^{-t} - e^{-3t}] = -e^{-t} + e^{-3t}$$

Steady state error,

$$e_{ss} = e(\infty) = -0 + 0 = 0$$

c) $r(t)$ is a unit ramp function $r(t) = t$ and $u(t) = 0$, then $R(s) = 1/s^2$ and $U(s) = 0$

$$C(s) = \frac{3}{s^2(s+1)(s+3)}$$

Partial Fraction Expansion,

$$C(s) = \frac{C_2}{s^2} + \frac{C_1}{s} + \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

$$C_2 = \lim_{s\to 0}\left[s^2\frac{3}{s^2(s+1)(s+3)}\right] = 1$$

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{3}{s^2(s+1)(s+3)}\right] = \frac{3}{2}$$

$$K_2 = \lim_{s\to -3}\left[(s+3)\frac{3}{s^2(s+1)(s+3)}\right] = -\frac{1}{6}$$

$$C(1)=\frac{3}{(1)(1+1)(1+3)}=\frac{1}{1}+\frac{C_1}{1}+\frac{3/2}{1+1}-\frac{1/6}{1+3} \qquad C_1=-\frac{4}{3}$$

$$C(s)=\frac{1}{s^2}-\frac{4/3}{s}+\frac{3/2}{s+1}-\frac{1/6}{s+3}$$

Transient response is obtained by Laplace Inverse:

$$c(t)=t-\frac{4}{3}+\frac{3}{2}e^{-t}-\frac{1}{6}e^{-3t}$$

Steady state error,

$$e_{ss}=\lim_{s\to 0}s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s)-\lim_{s\to 0}s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Since, $U(s)=0$ and $R(s)=1/s^2$

$$e_{ss}=\lim_{s\to 0}s\frac{1/s^2}{1+G_1(s)G_2(s)H(s)}=\lim_{s\to 0}\left[\frac{1}{s+sG_1(s)G_2(s)H(s)}\right]=\frac{1}{\lim_{s\to 0}sG_1(s)G_2(s)H(s)}$$

$$K_v=\lim_{s\to 0}sG_1(s)G_2(s)H(s)=\lim_{s\to 0}s\frac{3}{s(s+4)}=\frac{3}{4}$$

$$e_{ss}=\frac{1}{K_v}=\frac{1}{3/4}=\frac{4}{3}$$

Another way (for a unity feedback only)

$$e(t)=r(t)-c(t)$$

$$e(t)=t-\left[t-\frac{4}{3}+\frac{3}{2}e^{-t}-\frac{1}{6}e^{-3t}\right]=\frac{4}{3}-\frac{3}{2}e^{-t}+\frac{1}{6}e^{-3t}$$

Steady state error,

$$e_{ss}=e(\infty)=\frac{4}{3}-\frac{3}{2}e^{-(\infty)}+\frac{1}{6}e^{-3(\infty)}=\frac{4}{3}$$

d) Both $r(t)$ and $u(t)$ are unit step functions,

$$C(s)=\frac{3}{(s+1)(s+3)}R(s)+\frac{2s}{(s+1)(s+3)}U(s)$$

$$C(s)=\frac{3}{(s+1)(s+3)}\frac{1}{s}+\frac{2s}{(s+1)(s+3)}\frac{1}{s}$$

$$C(s)=\frac{3}{s(s+1)(s+3)}+\frac{2}{(s+1)(s+3)}$$

$$C(s)=\frac{3+2s}{s(s+1)(s+3)}$$

Using the techniques of Partial Fraction Expansion:

$$C(s)=\frac{K_1}{s}+\frac{K_2}{s+1}+\frac{K_3}{s+3}$$

$$K_1=\lim_{s\to 0}\left[s\frac{3+2s}{s(s+1)(s+3)}\right]=1$$

$$K_2 = \lim_{s \to -1}\left[(s+1)\frac{3+2s}{s(s+1)(s+3)}\right] = -\frac{1}{2}$$

$$K_3 = \lim_{s \to -3}\left[(s+3)\frac{3+2s}{s(s+1)(s+3)}\right] = -\frac{1}{2}$$

$$C(s) = \frac{1}{s} - \frac{1/2}{s+1} - \frac{1/2}{s+3}$$

Laplace Inverse,

$$c(t) = 1 - (1/2)e^{-t} - (1/2)e^{-3t}$$

Steady state error,

$$e_{ss} = \lim_{s \to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s) - \lim_{s \to 0} s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

Since, both unit step input $R(s) = U(t) = 1/s$

$$e_{ss} = \lim_{s \to 0}\frac{1}{1+G_1(s)G_2(s)H(s)} - \lim_{s \to 0}\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s \to 0} G_1(s)G_2(s)H(s)} - \frac{\lim_{s \to 0} G_2(s)H(s)}{1+\lim_{s \to 0} G_1(s)G_2(s)H(s)}$$

$$e_{ss} = \frac{1}{1+\lim_{s \to 0} G_1(s)G_2(s)H(s)} - \frac{\frac{1}{2}}{1+\lim_{s \to 0} G_1(s)G_2(s)H(s)} = \frac{\frac{1}{2}}{1+\lim_{s \to 0} G_1(s)G_2(s)H(s)}$$

$$K_p = \lim_{s \to 0} G_1(s)G_2(s)H(s) = \lim_{s \to 0}\frac{3}{s(s+4)} = \infty$$

$$e_{ss} = \frac{\frac{1}{2}}{1+\infty} = 0$$

Another way (for a unity feedback only)

$$e(t) = r(t) - c(t)$$

$$e(t) = 1 - [1 - (1/2)e^{-t} - (1/2)e^{-3t}] = (1/2)e^{-t} + (1/2)e^{-3t}$$

Steady state error,

$$e_{ss} = e(\infty) = (1/2)e^{-\infty} + (1/2)e^{-3\infty} = 0$$

e) $r(t)$ is a unit step function and $u(t)$ is a unit ramp function $u(t) = t$, then $R(s) = 1/s$ and $U(s) = 1/s^2$

$$C(s) = \frac{3}{(s+1)(s+3)}R(s) + \frac{2s}{(s+1)(s+3)}U(s)$$

$$C(s) = \frac{3}{(s+1)(s+3)}\frac{1}{s} + \frac{2s}{(s+1)(s+3)}\frac{1}{s^2}$$

$$C(s) = \frac{3}{s(s+1)(s+3)} + \frac{2}{s(s+1)(s+3)}$$

$$C(s) = \frac{5}{s(s+1)(s+3)}$$

Using the techniques of Partial Fraction Expansion:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+3}$$

$$K_1 = \lim_{s\to 0}\left[s\frac{5}{s(s+1)(s+3)}\right] = \frac{5}{3}$$

$$K_2 = \lim_{s\to -1}\left[(s+1)\frac{5}{s(s+1)(s+3)}\right] = -\frac{5}{2}$$

$$K_3 = \lim_{s\to -3}\left[(s+3)\frac{5}{s(s+1)(s+3)}\right] = \frac{5}{6}$$

$$C(s) = \frac{5/3}{s} - \frac{5/2}{s+1} + \frac{5/6}{s+3}$$

Laplace Inverse,

$$c(t) = \frac{5}{3} - \frac{5}{2}e^{-t} + \frac{5}{6}e^{-3t}$$

Steady state error,

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_1(s)G_2(s)H(s)}R(s) - \lim_{s\to 0} s\frac{G_2(s)H(s)}{1+G_1(s)G_2(s)H(s)}U(s)$$

$$e_{ss} = \lim_{s\to 0}\frac{1}{1+G_1(s)G_2(s)H(s)} - \lim_{s\to 0}\left[\frac{G_2(s)H(s)}{s+sG_1(s)G_2(s)H(s)}\right]$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0}G_1(s)G_2(s)H(s)} - \frac{\lim_{s\to 0}G_2(s)H(s)}{\lim_{s\to 0}sG_1(s)G_2(s)H(s)}$$

$$e_{ss} = \frac{1}{1+K_p} - \frac{1/2}{K_v}$$

$$K_p = \lim_{s\to 0}G_1(s)G_2(s)H(s) = \lim_{s\to 0}\frac{3}{s(s+4)} = \infty$$

$$K_v = \lim_{s\to 0}sG_1(s)G_2(s)H(s) = \lim_{s\to 0}s\frac{3}{s(s+4)} = \frac{3}{4}$$

$$e_{ss} = \frac{1}{1+\infty} - \frac{1/2}{3/4} = 0 - \frac{2}{7} = -\frac{2}{3}$$

Another way (for a unity feedback only)

$$e(t) = r(t) - c(t)$$

$$e(t) = 1 - \left[\frac{5}{3} - \frac{5}{2}e^{-t} + \frac{5}{6}e^{-3t}\right] = 1 - \frac{5}{3} + \frac{5}{2}e^{-t} - \frac{5}{6}e^{-3t}$$

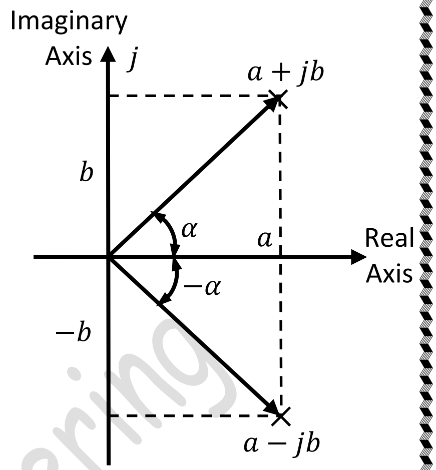
Steady state error,

$$e_{ss} = e(\infty) = 1 - \frac{5}{3} + \frac{5}{2}e^{-(\infty)} - \frac{5}{6}e^{-3(\infty)} = -\frac{2}{3}$$

Chapter 6

Laplace Transforms

Complex Conjugate Zeros:



Laplace transformed polynomial equation $B(s) = L_n(s)D_{F(s)}$ may have complex zeroes usually occur in pairs which conjugate each other. Complex conjugate zeros have same real parts and equal but opposite imaginary parts as shown in the **Figure**.

$$s_{1,2} = a \mp jb$$

$$B(s) = [s-(a+jb)][s-(a-jb)](s-r_1)(s-r_2)\cdots(s-r_{n-2})$$

Multiplication of complex conjugate zeros yields,

$$[s-(a+jb)][s-(a-jb)] = s^2 - 2as + a^2 + b^2$$

In which, is used to determine zeros for any quadratic equation.

For example,

$s^2 + 4s + 9 = s^2 - 2as + a^2 + b^2$

$4 = -2a$ $\qquad a = -2$

$9 = a^2 + b^2 = 4 + b^2$ $\qquad b = \mp\sqrt{5}$

$s_{1,2} = a \mp jb = -2 \mp j\sqrt{5}$ $\qquad$ complex conjugate zeros

Another example:

$s^2 + 8s + 12 = s^2 - 2as + a^2 + b^2$

$8 = -2a$ $\qquad a = -4$

$12 = a^2 + b^2 = 16 + b^2$ $\qquad b = \mp j2$

$s_{1,2} = a \mp jb = -4 \mp j(j2)$ $\qquad s_{1,2} = a \mp jb = -2, -6$ $\qquad$ Real zeros

Laplace transforms equation of control system,

$$Y(s) = \frac{A(s)}{[s-(a+jb)][s-(a-jb)](s-r_1)(s-r_2)\cdots(s-r_{n-2})}$$

Using techniques of Partial Fraction Expansion yields:

$$Y(s) = \frac{K_c}{[s-(a+jb)]} + \frac{K_{-c}}{[s-(a-jb)]} + \frac{K_1}{s-r_1} + \frac{K_2}{s-r_2} + \cdots\cdots + \frac{K_{n-2}}{s-r_{n-2}}$$

Transient response Laplace Inverse (general form),

$$y(t) = K_c e^{(a+jb)t} + K_{-c} e^{(a-jb)t} + K_1 e^{r_1 t} + K_2 e^{r_2 t} + \cdots\cdots + K_{n-2} e^{r_{n-2} t}$$

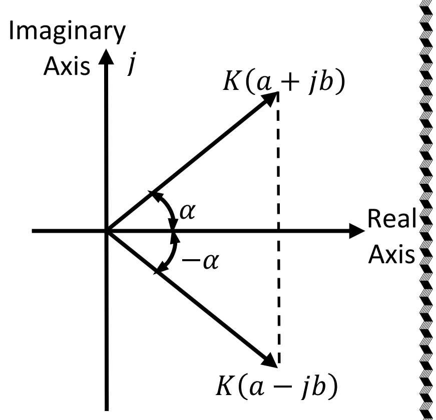
This can be expressed as,

$$y(t) = \frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha) + K_1 e^{r_1 t} + K_2 e^{r_2 t} + \cdots\cdots + K_{n-2} e^{r_{n-2} t}$$

Where,

$$K(a+jb) = [(s^2 - 2as + a^2 + b^2)Y(s)]_{s=a+jb}$$

$$K(a-jb) = [(s^2 - 2as + a^2 + b^2)Y(s)]_{s=a-jb}$$



Constants $K(a+jb)$ and its conjugate $K(a-jb)$ are complex numbers that have magnitude and direction as shown in **Figure**.

$$|K(a+jb)| = \sqrt{[\text{Re. of } K(a+jb)]^2 + [\text{Im. of } K(a+jb)]^2}$$

$$\alpha = \tan^{-1}\frac{\text{Im. of } K(a+jb)}{\text{Re. of } K(a+jb)}$$

Example 1:

Determine the transient response or the inverse transformation of the following transformed equation:

$$Y(s) = \frac{75}{(s^2+4s+13)(s+6)}$$

Solution:

$$s^2+4s+13 = s^2-2as+a^2+b^2$$

$$4=-2as \qquad\qquad a=-2$$

$$13=a^2+b^2 \qquad\qquad b=\mp 3$$

$$s_{1,2}=a\mp jb=-2\mp j3 \qquad \text{complex conjugate zeros}$$

$$Y(s)=\frac{75}{[s-(-2+j3)][s-(-2-j3)](s+6)}$$

Using techniques of Partial Fraction Expansion yields:

$$Y(s)=\frac{K_c}{[s-(-2+j3)]}+\frac{K_{-c}}{[s-(-2-j3)]}+\frac{K_1}{s+6}$$

Where,

$$K_1=\lim_{s\to -6}\left[(s+6)\frac{75}{(s^2+4s+13)(s+6)}\right]=3$$

Laplace Inverse yields:

$$y(t)=K_c e^{(-2+j3)t}+K_{-c}e^{(-2-j3)t}+3e^{-6t}$$

Transient response could be expressed as:

$$y(t)=\frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha)+K_1e^{r_1t}$$

$$y(t)=\frac{1}{3}|K(a+jb)|e^{-2t}\sin(3t+\alpha)+3e^{-6t}$$

$$K(a+jb)=[(s^2-2as+a^2+b^2)Y(s)]_{s=a+bj}$$

$$K(a+jb)=\left[(s^2+4s+13)\frac{75}{(s^2+4s+13)(s+6)}\right]_{s=-2+j3}$$

$$K(a+jb)=\frac{75}{4+j3}$$

Magnitude of $K(a+jb)$,

$$|K(a+jb)|=\left|\frac{75}{4+j3}\right|=\frac{|75|}{|4+j3|}=\frac{75}{\sqrt{4^2+3^2}}=15$$

Angle α is obtained as:

$$\alpha=\sphericalangle K(a+jb)=\sphericalangle\frac{75}{4+j3}=\sphericalangle(75)-\sphericalangle(4+j3)=\tan^{-1}\frac{0}{75}-\tan^{-1}\frac{3}{4}=0-36.8=-36.8°$$

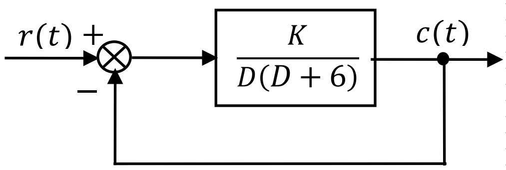
Transient response or time solution of the control system,

$$y(t)=5e^{-2t}\sin(3t-36.8°)+3e^{-6t}$$

The term $[5e^{-2t}\sin(3t-36.8)]$ is an exponentially damped sinusoidal term introduced by a pair of complex conjugate zeros.

Example 2:

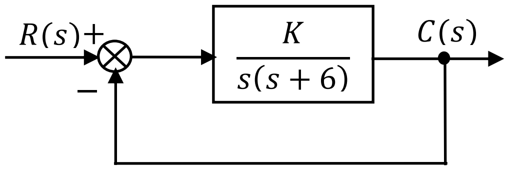
The block-diagram representation of a hydraulic system which provides the power for a numerically controlled machine tool is shown in **Figure**. The forcing function is $r(t)=u(t)$ and all the initial conditions are zero. Determine the response $c(t)$ and the steady-state error for a unit step input when,



$(a)\ K = 5$ $(b)\ K = 9$ $(c)\ K = 25$

Solution:

Since all the initial conditions are zero, the block-diagram could be represented by Laplace transforms as shown in the Figure. Transfer function is,



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+6)}}{1+\frac{K}{s(s+6)}} = \frac{K}{s(s+6)+K}$$

$$C(s) = \frac{K}{s^2+6s+K}R(s)$$

Since $r(t) = u(t),$ then $R(s) = \frac{1}{s}$

$$C(s) = \frac{K}{s(s^2+6s+K)}$$

Steady-state error for a unit step input is determined as:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0}\frac{K}{s(s+6)} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

a) For, $K = 5$

$$C(s) = \frac{5}{s(s^2+6s+5)} = \frac{5}{s(s+1)(s+5)}$$

Since zeros are distinct, using techniques of Partial-Fraction-Expansion:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+5}$$

$$K_1 = \lim_{s\to 0}\left[s\frac{5}{s(s+1)(s+5)}\right] = 1$$

$$K_2 = \lim_{s\to -1}\left[(s+1)\frac{5}{s(s+1)(s+5)}\right] = -\frac{5}{4}$$

$$K_3 = \lim_{s\to -5}\left[(s+5)\frac{5}{s(s+1)(s+5)}\right] = \frac{1}{4}$$

$$C(s) = \frac{1}{s} - \frac{5/4}{s+1} + \frac{1/4}{s+5}$$

Laplace inverse of transformed equation yields transient response,

$$c(t) = 1 - \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$$

b) For, $K = 9$

$$C(s) = \frac{9}{s(s^2+6s+9)} = \frac{9}{s(s+3)^2}$$

Since zeros are distinct, using techniques of Partial-Fraction-Expansion:

$$C(s) = \frac{C_2}{(s+3)^2} + \frac{C_1}{s+3} + \frac{K_1}{s}$$

$$C_2 = \lim_{s\to -3}\left[(s+3)^2\frac{9}{s(s+3)^2}\right] = -3$$

$$K_1 = \lim_{s\to 0}\left[s\frac{9}{s(s+3)^2}\right] = 1$$

$$C(s) = \frac{9}{s(s+3)^2} = -\frac{3}{(s+3)^2} + \frac{C_1}{s+3} + \frac{1}{s}$$

For, $s = 1$

$$C(1) = \frac{9}{(1)(1+3)^2} = -\frac{3}{(1+3)^2} + \frac{C_1}{1+3} + \frac{1}{1} \qquad C_1 = -1$$

$$C(s) = -\frac{3}{(s+3)^2} - \frac{1}{s+3} + \frac{1}{s}$$

Laplace inverse of transformed equation yields transient response,

$$c(t) = \left[\frac{C_q t^{q-1}}{(q-1)!} + \frac{C_{q-1} t^{q-2}}{(q-2)!} + \cdots\cdots + \frac{C_2 t}{1!} + C_1\right] e^{rt} + K_1 e^{r_1 t} + \cdots\cdots + K_{n-q} e^{r_{n-q} t}$$

$$c(t) = [-3t - 1]e^{-3t} + 1$$

c) For, $K = 25$

$$C(s) = \frac{25}{s(s^2 + 6s + 25)}$$

$s^2 + 6s + 25 = s^2 - 2as + a^2 + b^2$
$6 = -2a$ $\qquad a = -3$
$25 = a^2 + b^2 = 9 + b^2$ $\qquad b = \mp 4$
$s_{1,2} = a \mp jb = -3 \mp j4$ $\qquad$ complex conjugate zeros

$$C(s) = \frac{25}{s[s - (-3 + j4)][s - (-3 - j4)]}$$

Since zeros are complex conjugate, using techniques of Partial-Fraction-Expansion:

$$C(s) = \frac{K_c}{[s - (-3 + j4)]} + \frac{K_{-c}}{[s - (-3 - j4)]} + \frac{K_1}{s}$$

Where,

$$K_1 = \lim_{s \to 0}\left[s \frac{25}{s(s^2 + 6s + 25)}\right] = 1$$

Laplace Inverse yields:

$$c(t) = K_c e^{(-3+j4)t} + K_{-c} e^{(-3-j4)t} + K_1 e^{r_1 t}$$

Transient response could be expressed as:

$$c(t) = \frac{1}{b}|K(a + jb)| e^{at} \sin(bt + \alpha) + K_1 e^{r_1 t}$$

$$c(t) = \frac{1}{4}|K(a + jb)| e^{-3t} \sin(4t + \alpha) + 1$$

$$K(a + jb) = [(s^2 - 2as + a^2 + b^2)C(s)]_{s=a+bj}$$

$$K(a + jb) = \left[(s^2 + 6s + 25)\frac{25}{s(s^2 + 6s + 25)}\right]_{s=-3+j4} = \frac{25}{-3 + j4}$$

Magnitude of $K(a + jb)$,

$$|K(a + jb)| = \left|\frac{25}{-3 + j4}\right| = \frac{|25|}{|-3 + j4|} = \frac{25}{\sqrt{(-3)^2 + 4^2}} = 5$$

Angle α is obtained as:

$$\alpha = \sphericalangle K(a + jb) = \sphericalangle \frac{25}{-3 + j4} = \sphericalangle(25) - \sphericalangle(-3 + j4) = \tan^{-1}\frac{0}{25} - \tan^{-1}\frac{-3}{4} = 0 - (-36.8)$$
$$= 36.8°$$

Transient response or time solution of the control system,

$$c(t) = \frac{5}{4} e^{-3t} \sin(4t + 36.8°) + 1$$

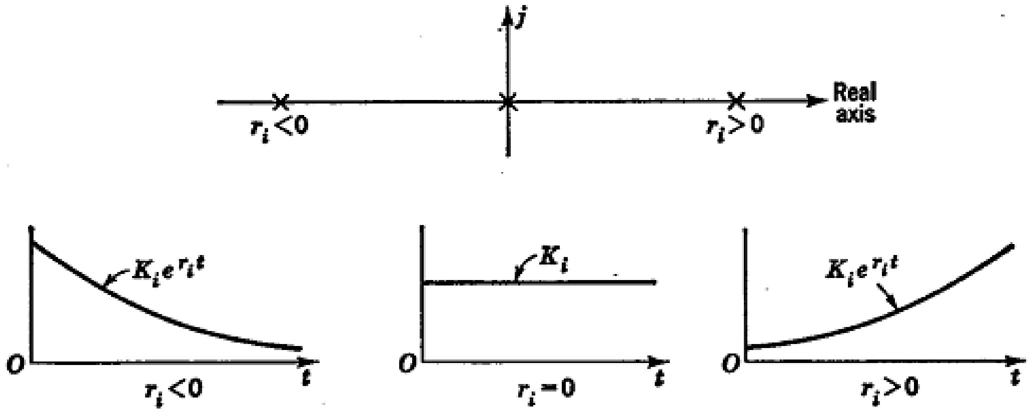
The term $\left[\frac{5}{4}e^{-3t}\sin(4t + 36.8°)\right]$ is an exponentially damped sinusoidal term introduced by a pair of complex conjugate zeros.

Transient Response:

A very good measure of transient response or transient behavior may be obtained directly from location of zeros of $B(s) = L_n(s)D_{F(s)}$ on s-plane. General form of transient response in the case of real zeros is:

$$y(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} \ldots\ldots + K_n e^{r_n t}$$

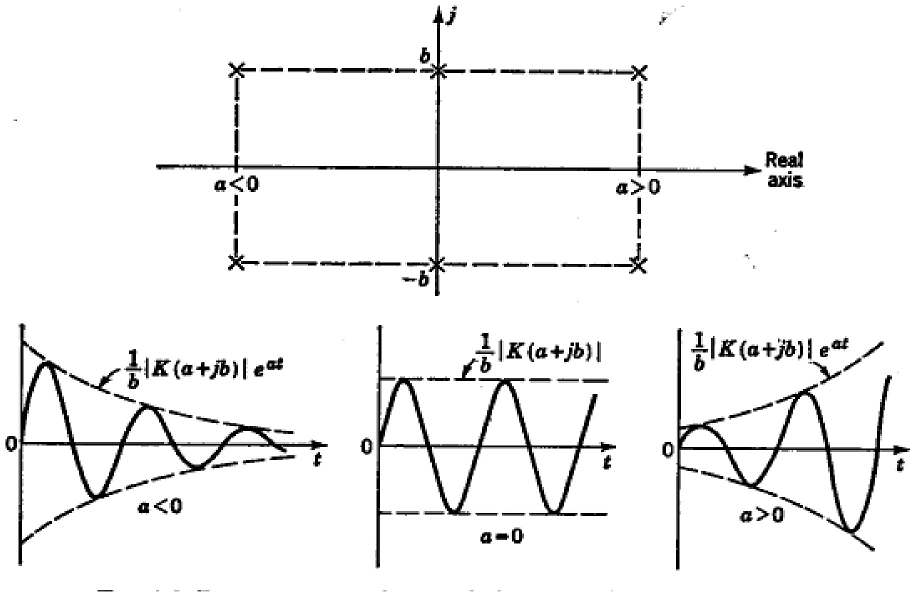
A negative zero $r_i < 0$ located on left of imaginary axis yields an exponentially decreasing term while a positive zero $r_i > 0$ located on right of imaginary axis yields an exponentially increasing term. A zero at the origin $r_i = 0$ results a constant term as shown,



Transient response of a control system with complex conjugate zeros is:

$$y(t) = \frac{1}{b}|K(a + jb)|e^{at}\sin(bt + \alpha) + K_1 e^{r_1 t} + K_2 e^{r_2 t} + \cdots + K_{n-2} e^{r_{n-2} t}$$

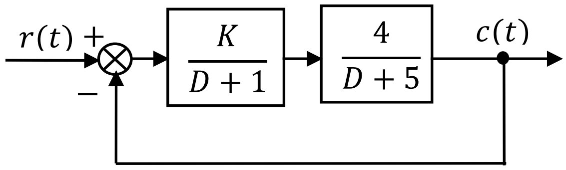
An exponentially damped sinusoidal term may be existed from complex conjugate zeros of $B(s)$. Real part a of complex conjugate zero is exponential factor, when zeros lie to left of imaginary axis $a < 0$ a decreasing sinusoidal term results while an increasing sinusoidal term results when zeros are located to the right of imaginary axis as shown in **Figure**. When zeros are located on the imaginary axis, $a = 0$ a sinusoid of constant amplitude results.



Example 3:

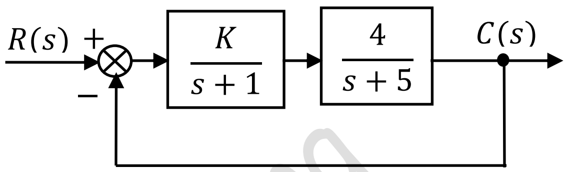
For the system shown in Figure, the forcing function is $r(t) = e^{-t}$ and all of the initial conditions are zero. Determine the response $c(t)$ and steady-state error for each of the following cases:

$(a)\ K = 1$ (Home Work) $(b)\ K = 2$

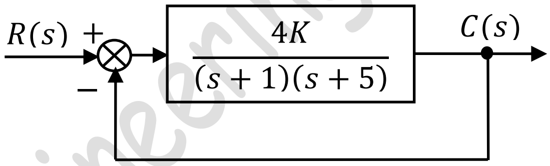


Solution:

Since all the initial conditions are zero, block-diagram could be represented by Laplace transforms as shown in **Figure**.



Using the rule of combining blocks in cascade yields, Transfer function is,



$$\frac{C(s)}{R(s)} = \frac{\frac{4K}{(s+1)(s+5)}}{1+\frac{4K}{(s+1)(s+5)}} = \frac{4K}{s^2+6s+5+4K}$$

$$C(s) = \frac{4K}{s^2+6s+5+4K}R(s)$$

$(b)\ K = 2,$

$$C(s) = \frac{8}{s^2+6s+13}R(s)$$

Since, $r(t) = e^{-t}$ then $R(s) = \frac{1}{s+1}$

$$C(s) = \frac{8}{(s+1)(s^2+6s+13)}$$

$s^2+6s+13 = s^2-2as+a^2+b^2$

$6 = -2a$ $a = -3$

$13 = a^2+b^2 = 9+b^2$ $b = \mp 2$

$s_{1,2} = a \mp jb = -3 \mp j2$ complex conjugate zeros

$$C(s) = \frac{8}{[s-(-3+j2)][s-(-3-j2)](s+1)}$$

Using techniques of Partial-Fraction-Expansion:

$$C(s) = \frac{K_c}{[s-(-3+j2)]} + \frac{K_{-c}}{[s-(-3-j2)]} + \frac{K_1}{s+1}$$

Where,

$$K_1 = \lim_{s\to -1}\left[(s+1)\frac{8}{(s+1)(s^2+6s+13)}\right] = 1$$

Laplace Inverse yields:

$$c(t) = K_c e^{(-3+j2)t} + K_{-c}e^{(-3-j2)t} + e^{-t}$$

Transient response could be expressed as:

$$c(t) = \frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha) + K_1 e^{r_1 t}$$

$$c(t) = \frac{1}{2}|K(a+jb)|e^{-3t}\sin(2t+\alpha) + e^{-t}$$

$$K(a+jb) = [(s^2-2as+a^2+b^2)C(s)]_{s=a+jb}$$

$$K(a+jb) = \left[(s^2+6s+13)\frac{8}{(s+1)(s^2+6s+13)}\right]_{s=-3+j2} = \frac{8}{-2+j2}$$

Magnitude of $K(a+jb)$,

$$|K(a+jb)| = \left|\frac{8}{-2+j2}\right| = \frac{|8|}{|-2+j2|} = \frac{8}{\sqrt{(-2)^2+2^2}} = \frac{8}{\sqrt{8}} = 2\sqrt{2}$$

Angle α is obtained as:

$$\alpha = \measuredangle K(a+jb) = \measuredangle\frac{8}{-2+j2} = \measuredangle(8) - \measuredangle(-2+j2) = \tan^{-1}\frac{0}{8} - \tan^{-1}\frac{2}{-2} = 0-(-45) = 45°$$

Transient response or time solution of the control system,

$$c(t) = \sqrt{2}e^{-3t}\sin(2t+45°) + e^{-t}$$

The term $\left[\sqrt{2}e^{-3t}\sin(2t+45°)\right]$ is an exponentially damped sinusoidal term introduced by a pair of complex conjugate zeros.

The steady-state error could be determined from transient or dynamic response:

$$e(t) = r(t) - c(t)$$

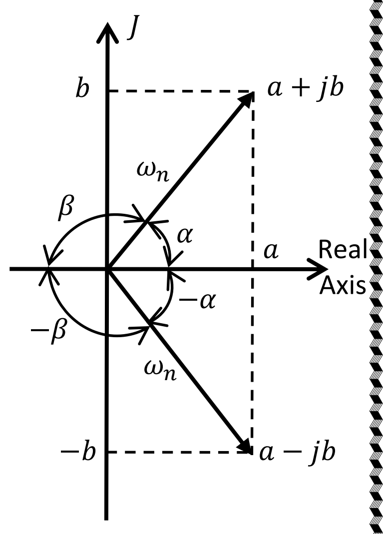
$$e(t) = e^{-t} - [\sqrt{2}e^{-3t}\sin(2t+45°) + e^{-t}]$$

$$e(t) = \sqrt{2}e^{-3t}\sin(2t+45°)$$

Steady state error,

$$e_{ss} = e(\infty) = \sqrt{2}e^{-3(\infty)}\sin(2(\infty)+45°) = 0$$

Damping Ratio and Natural Frequency:



Since a pair complex conjugate zeros introduces an exponentially damping sinusoidal term, it can be expressed in terms of damping ratio ξ and natural frequency ω_n rather than a and b.

$[s-(a+jb)][s-(a-jb)] = s^2 - 2as + a^2 + b^2$

$\omega_n = \sqrt{a^2+b^2}$ Undamped natural frequency

$a = \omega_n\cos\alpha = \omega_n\cos(\pi-\beta) = -\omega_n\cos\beta$

$[s-(a+jb)][s-(a-jb)] = s^2 - 2as + a^2 + b^2 = s^2 + 2\omega_n\cos\beta\, s + \omega_n^2$

Since,

$-2as = 2\omega_n\cos\beta$ Actual amount of damping

For critical damping $\beta = 0$

$-2a = 2\omega_n\cos\beta = 2\omega_n$ Critical damping

Damping ratio is the ratio of actual amount of damping to critical damping.

$$\xi = \frac{2\omega_n\cos\beta}{2\omega_n} = \cos\beta$$

Complex conjugate zeros can be expressed in terms ξ and ω_n as,

$$[s-(a+jb)][s-(a-jb)] = s^2 + 2\xi\omega_n s + \omega_n^2$$

A pair of complex conjugate zeros can be specified in ξ and ω_n as:

$$s_{1,2} = -\xi\omega_n \mp j\omega_n\sqrt{1-\xi^2}$$

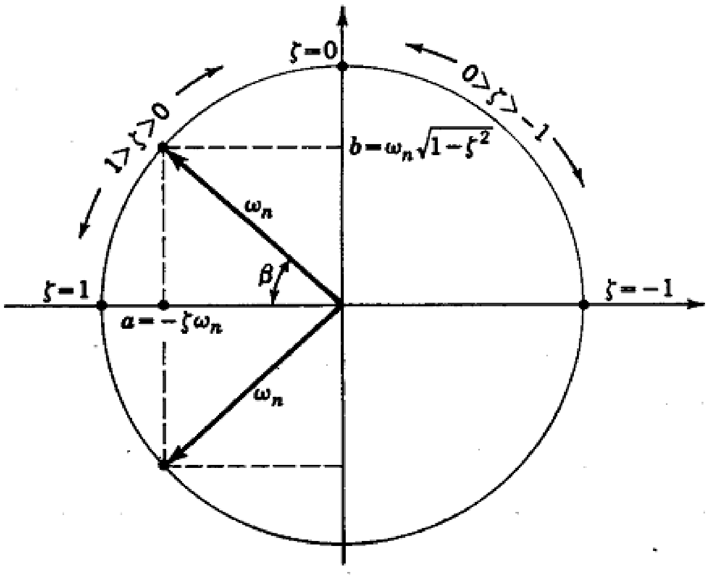
$b = \mp\sqrt{\omega_n^2 - a^2} = \mp\omega_n\sqrt{1-\cos\beta^2} = \mp\omega_n\sqrt{1-\xi^2} = \omega_d$ Damped natural frequency

Transient response is expressed in terms of damping ratio ξ and natural frequency, (ω_n):

$$y(t) = \frac{1}{\omega_n\sqrt{1-\xi^2}}|K(a+jb)|e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right) + K_1e^{r_1t}\cdots + K_{n-2}e^{r_{n-2}t}$$

$$-1 < \xi < 1$$

Note, when the damping ratio ξ is greater than one, the zeros are no longer complex conjugate but are real. The response is no longer sinusoidal but is exponential. Consider the following general plot of a pair of complex conjugate roots:



1. When $a < 0$ a decreasing exponential sinusoidal term results and the roots lie to the left side of imaginary axis where $0 < \beta < 90°$ in which $1 > \xi > 0$ Positive value of ξ yields a decreasing sinusoidal response term.
2. When $a > 0$ an increasing exponential sinusoidal term results and the roots lie to the right side of imaginary axis where $90° < \beta < 180°$ in which $0 > \xi > -1$ Negative value of ξ yields an increasing sinusoidal response term.
3. When $a = 0$ a sinusoidal term of constant amplitude results and the roots lie on the imaginary axis where $\beta = 90°$ in which $\xi = 0$ zero value of ξ yields a sinusoidal term of constant amplitude.

Example 4:

Determine the general equation for the transient response of a second-order system to a unit step function change which occurs at $t = 0$. The operational form of the differential equation is:

$$y(t) = \frac{\omega_n^2}{D^2 + 2\xi\omega_n D + \omega_n^2} f(t)$$

Assume that all the initial conditions are zero. Then solve for $\xi = 1$?

Solution:

Laplace transform,

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A(s)}{B(s)}$$

$$Y(s) = \frac{\omega_n^2}{s\left[s - \left(-\xi\omega_n + \omega_n\sqrt{1-\xi^2}\right)\right]\left[s - \left(-\xi\omega_n - \omega_n\sqrt{1-\xi^2}\right)\right]}$$

Techniques of Partial Fraction Expansion

$$Y(s) = \frac{K_c}{s - \left(-\xi\omega_n + \omega_n\sqrt{1-\xi^2}\right)} + \frac{K_{-c}}{s - \left(-\xi\omega_n - \omega_n\sqrt{1-\xi^2}\right)} + \frac{K_1}{s}$$

Transient response for control system with complex conjugate zeros could be expressed as:

$$y(t) = \frac{1}{b}|K(a + jb)|e^{at}\sin(bt + \alpha) + K_1 e^{r_1 t}$$

Or,

$$y(t)=\frac{1}{\omega_n\sqrt{1-\xi^2}}|K(a+jb)|e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right)+K_1e^{r_1t}$$

$$K(a+jb)=\left[(s^2-2as+a^2+b^2)\frac{A(s)}{B(s)}\right]_{s=a+bj}$$

$$K(a+jb)=\left[(s^2+2\xi\omega_n s+\omega_n^2)\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}\right]_{s=-\xi\omega_n+\omega_n\sqrt{1-\xi^2}}$$

$$K(a+jb)=\frac{\omega_n^2}{-\xi\omega_n+\omega_n\sqrt{1-\xi^2}}=\frac{\omega_n}{-\xi+j\sqrt{1-\xi^2}}$$

$$K(a+jb)=\frac{\omega_n}{-\xi+j\sqrt{1-\xi^2}}\times\frac{-\xi-j\sqrt{1-\xi^2}}{-\xi-j\sqrt{1-\xi^2}}=-\xi\omega_n-j\omega_n\sqrt{1-\xi^2}$$

$$|K(a+jb)|=\left|-\xi\omega_n-j\omega_n\sqrt{1-\xi^2}\right|=\omega_n$$

$$\alpha=\tan^{-1}\frac{Imaginary\ part\ of\ K(a+jb)}{Real\ part\ of\ K(a+jb)}$$

$$\alpha=tan^{-1}\frac{-\omega_n\sqrt{1-\xi^2}}{-\xi\omega_n}=tan^{-1}\frac{-\sqrt{1-\xi^2}}{-\xi}=tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

$$K_1=\lim_{s\to 0}\left[s\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}\right]=1$$

$$y(t)=\frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha)+K_1e^{r_1t}$$

$$y(t)=\frac{1}{\omega_n\sqrt{1-\xi^2}}\omega_n e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right)+1$$

The general equation for the transient response:

$$y(t)=\frac{1}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right)+1$$

This transient response equation is valid for $-1<\xi<1$

For $\xi=1$,

$$Y(s)=\frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{s(s+\omega_n)^2}$$

Using the techniques of Partial Fraction Expansion:

$$Y(s)=\frac{C_2}{(s+\omega_n)^2}+\frac{C_1}{s+\omega_n}+\frac{K_1}{s}$$

$$C_2=\lim_{s\to-\omega_n}\left[(s+\omega_n)^2\frac{\omega_n^2}{s(s+\omega_n)^2}\right]=-\omega_n$$

$$C_1=\lim_{s\to-\omega_n}\frac{-\omega_n^2}{s^2}=-1$$

$$K_1=\lim_{s\to 0}\left[s\frac{\omega_n^2}{s(s+\omega_n)^2}\right]=1$$

$$Y(s)=\frac{\omega_n^2}{s(s+\omega_n)^2}=\frac{-\omega_n}{(s+\omega_n)^2}+\frac{C_1}{s+\omega_n}+\frac{1}{s}$$

$$Y(1)=\frac{\omega_n^2}{(1)(1+\omega_n)^2}=\frac{-\omega_n}{(1+\omega_n)^2}+\frac{C_1}{1+\omega_n}+\frac{1}{1}$$

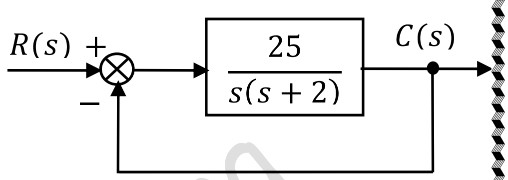
$$Y(s)=\frac{-\omega_n}{(s+\omega_n)^2}-\frac{1}{s+\omega_n}+\frac{1}{s}$$

Laplace inverse yields:

$$y(t)=1-(\omega_n t+1)e^{-\omega_n t}$$

Example 5:

Determine the damping ratio ξ , undamped natural frequency ω_n , and damped natural frequency for the system shown in Figure. What is the response $c(t)$ of this system to a unit step function excitation $r(t)=u(t)$ when initial conditions are zero ?



Solution:

Transfer function is,

$$\frac{C(s)}{R(s)}=\frac{\frac{25}{s(s+2)}}{1+\frac{25}{s(s+2)}}=\frac{25}{s^2+2s+25}$$

$$C(s)=\frac{25}{s^2+2s+25}R(s)$$

Since, $r(t)=u(t)$ then $R(s)=\frac{1}{s}$

$$C(s)=\frac{25}{s(s^2+2s+25)}$$

$s^2+2s+25=s^2+2\xi\omega_n s+\omega_n^2$

$25=\omega_n^2$ $\omega_n=5$ Undamped natural frequency

$2=2\xi\omega_n$ $\xi=0.2$ Damping ratio

$b=\mp\omega_n\sqrt{1-\xi^2}=\mp 5\sqrt{1-(0.2)^2}=\mp 2\sqrt{6}$ Damped natural frequency

$s^2+2s+25=s^2+2\xi\omega_n s+\omega_n^2$

Transient response is expressed in terms of damping ratio ξ and natural frequency ω_n:

$$y(t)=\frac{1}{\omega_n\sqrt{1-\xi^2}}|K(a+jb)|e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right)+K_1e^{r_1t}\cdots+K_{n-2}e^{r_{n-2}t}$$

$$-1<\xi<1$$

Since, $a=-\xi\omega_n$ and $b=\omega_n\sqrt{1-\xi^2}$

$$K(a+jb)=\frac{\omega_n^2}{-\xi\omega_n+j\omega_n\sqrt{1-\xi^2}}=\frac{\omega_n}{-\xi+j\sqrt{1-\xi^2}}$$

$$K(a+jb)=\frac{\omega_n}{-\xi+j\sqrt{1-\xi^2}}\times\frac{-\xi-j\sqrt{1-\xi^2}}{-\xi-j\sqrt{1-\xi^2}}=-\xi\omega_n-j\omega_n\sqrt{1-\xi^2}$$

$$|K(a+jb)|=\left|-\xi\omega_n-j\omega_n\sqrt{1-\xi^2}\right|=\omega_n$$

$$\alpha=\tan^{-1}\frac{Imaginary\ part\ of\ K(a+jb)}{Real\ part\ of\ K(a+jb)}$$

$$\alpha=tan^{-1}\frac{-\omega_n\sqrt{1-\xi^2}}{-\xi\omega_n}=tan^{-1}\frac{-\sqrt{1-\xi^2}}{-\xi}=tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

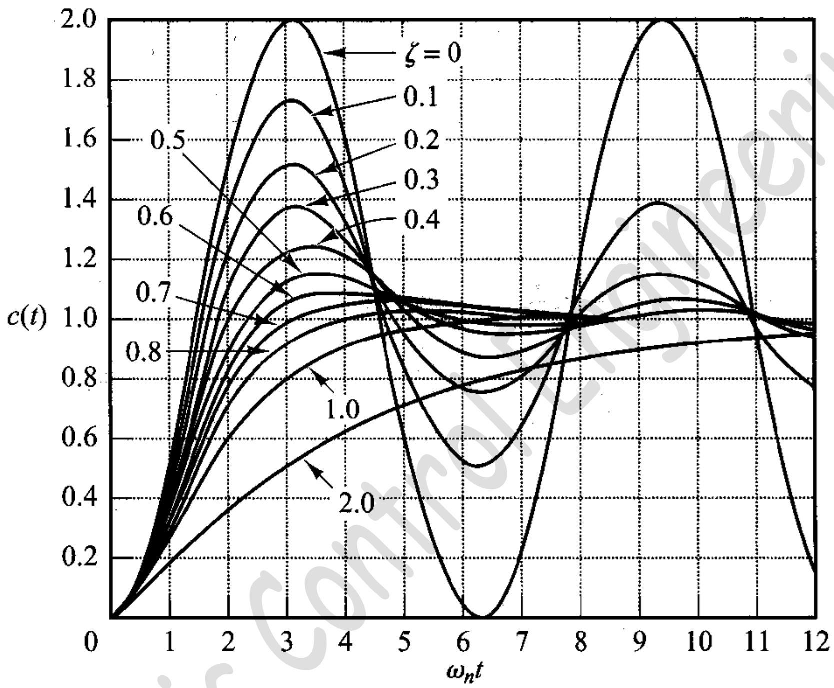
$$K_1 = \lim_{s\to 0}\left[s\frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}\right] = 1$$

$$y(t) = \frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha) + K_1 e^{r_1 t}$$

$$y(t) = \frac{1}{\omega_n\sqrt{1-\xi^2}}\omega_n e^{-\xi\omega_n t}\sin\left(\omega_n\sqrt{1-\xi^2}t+\alpha\right) + 1$$

Influence of damping ratio ξ on transient response:

The response of a second-order system to a unit step function for various values of the damping ratio ξ



Transient Response Specifications:

A typical response of a second-order control system to a step input function is shown in Figure:

- Rise time t_r is the time at which the response first attains its final steady-state value. Thus final steady-state value occurs firstly at rise time t_r.

$$t_r = \frac{tan^{-1}\left(\sqrt{1-\xi^2}/-\xi\right)}{\omega_d} = \frac{tan^{-1}\left(\sqrt{1-\xi^2}/-\xi\right)}{\omega_n\sqrt{1-\xi^2}}$$

- Peak time t_p is the time at which the response reaches its maximum overshoot. Maximum value of $y(t)$ occurs when:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

- Maximum percent overshoot is:

$$PO = 100\frac{y_{max} - h}{h} = 100e^{\left(-\xi\pi/\sqrt{1-\xi^2}\right)}$$

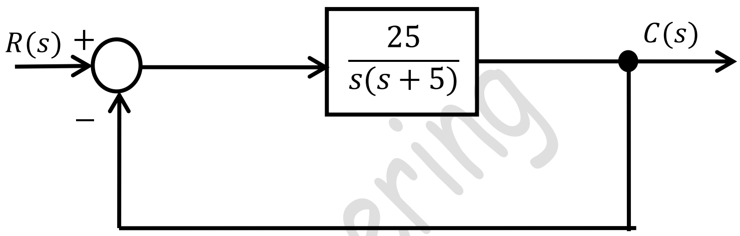
- settling time t_s is the time required before the response does not oscillate more than some small percentage such as 2 Or 5 percent from the final steady-state value.

$$t_s = \frac{3}{\xi\omega_n} \qquad for\ 5\%$$

$$t_s = \frac{4}{\xi\omega_n} \qquad for\ 2\%$$

Example 6:

For the feedback control system shown in figure, determine the natural frequency, damping ratio, damped natural frequency, rise time, peak time, percent overshoot, and approximate 5 percent settling time.



Solution:

$$\frac{C(s)}{R(s)} = \frac{\frac{25}{s(s+5)}}{1 + \frac{25}{s(s+5)}} = \frac{25}{s^2 + 5s + 25}$$

$$C(s) = \frac{25}{s^2 + 5s + 25}R(s)$$

$$s^2 + 5s + 25 = s^2 + 2\xi\omega_n s + \omega_n^2$$

In which:

$$\omega_n^2 = 25 \qquad and \qquad \omega_n = 5$$

$$2\xi\omega_n = 5 \qquad and \qquad \xi = 0.5$$

- Natural frequency $\omega_n = 5$
- Damping ratio $\xi = 0.5$
- Damped natural frequency:

$$\omega_d = b = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.5)^2} = 4.33\ rad/s$$

- Rise time,

$$t_r = \frac{tan^{-1}\left(\sqrt{1-\xi^2}/-\xi\right)}{\omega_d} = \frac{tan^{-1}\left(\sqrt{1-(0.5)^2}/-0.5\right)}{4.33} = 0.483\ s$$

- Peak time,

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.33} = 0.725\ s$$

- Percent overshoot,

$$PO = 100e^{\left(-\xi\pi/\sqrt{1-\xi^2}\right)} = 100e^{\left(-(0.5)\pi/\sqrt{1-(0.5)^2}\right)} = 16.3\%$$

- Approximate 5% settling time,

$$t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.5(5)} = 1.2\ s$$

Routh's Stability Criterion:

There is a major difficulty to determine the roots of the characteristic equation of a feedback control system. Routh's stability criterion is a method for determining whether or not any of the roots of the characteristic equation are in the right half-plane.

- Write the characteristic equation in the general form:

$$b_n s^n + b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + \cdots\cdots + b_2 s^2 + b_1 s + b_o = 0$$

- Arrange the coefficients of the characteristic equation as:

b_n	b_{n-2}	b_{n-4}	b_{n-6}	$\cdots$
b_{n-1}	b_{n-3}	b_{n-5}	b_{n-7}	$\cdots$
c_1	c_2	c_3	c_4	$\cdots$
d_1	d_2	d_3	$\cdots$	
$\dots\dots$	$\dots\dots$	$\dots\dots$		
e_1	e_2	0		
f_1	f_2	0		
g_1	0			
h	0			

- The row of c is evaluated as:

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}}$$
$$c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}$$
$$c_3 = \frac{b_{n-1}b_{n-6} - b_n b_{n-7}}{b_{n-1}}$$

- The coefficient d is obtained as:

$$d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1}$$
$$d_2 = \frac{c_1 b_{n-5} - b_{n-1} c_3}{c_1}$$

- Process should be continued until one more row is obtained than the order of the differential equation.

Example 7:

Consider the characteristic function and examine the stability

$$s^4 + 3s^3 + s^2 + 6s + 2 = 0$$

Solution:

s^4	1	1	2	0
s^3	3	6	0	
s^2	-1	2	0	
s^1	12	0		
s^0	2	0		

Since the number of changes of sign of the coefficients in the first row is two then two roots located to the right of the imaginary axis.

A zero in the First Column:

When one of the coefficients in the first column is zero, it may be replaced by a very small number ε for the purpose of computing the remaining coefficients in array.

Example 8:

Consider the characteristic function and examine the stability

$$s^5 + 2s^4 + 4s^3 + 8s^2 + 10s + 6 = 0$$

Solution:

s^5	1	4	10	0
s^4	2	8	6	0
s^3	$0 \approx \varepsilon$	7	0	
s^2	$8 - 14/\varepsilon$	6	0	
s^1	7	0		
s^0	6	0		

Since $8 - 14/\varepsilon$ is a very large negative number when ε is positive and is a very large positive number when ε is negative. Therefore two sign changes in first column and characteristic equation has two roots located to the right of imaginary axis.

A row of zeros:

A row of zeros occurs in Routh's array means that the characteristic equation has:

- A pair of real roots with opposite signs.
- Complex conjugate roots on the imaginary axis.
- A pair of complex conjugate roots with opposite real parts.

The following procedure should be implemented:

- Auxiliary equation $A(s)$ is formed using coefficients of the row above row of zeros.
- Derive the auxiliary equation with respect to s and replace the coefficients of row of zeros by coefficients of the derived auxiliary equation.
- Continue solving according to the Routh's Criterion techniques.

Example 9:

Determine the stability of the control system whose characteristic equation is:

$$s^6 + 6s^5 + 10s^4 + 12s^3 + 13s^2 - 18s - 24 = 0$$

Solution:

s^6	1	10	13	-24	0
s^5	6	12	-18	0	
s^4	8	16	-24	0	
s^3	0	0	0		

Auxiliary equation is formed from the row above the row of zeros,

$A(s) = 8s^4 + 16s^2 - 24$ $\qquad \frac{dA(s)}{ds} = 32s^3 + 32s$

s^6	1	10	13	-24	0

s^5	6	12	−18	0
s^4	8	16	−24	0
s^3	32	32	0	
s^2	8	−24	0	
s^1	128	0		
s^0	−24			

One sign changes from positive to negative, therefore there is one root of the characteristic equation is located in right half-plane and system is unstable.

Example 10:

Determine the stability of the control system whose characteristic equation is:

$$s^6 + 3s^5 + 2s^4 + 4s^2 + 12s + 8 = 0$$

Solution:

s^6	1	2	4	8	0
s^5	3	0	12	0	
s^4	2	0	8	0	
s^3	0	0	0		

Auxiliary equation is formed from the row above the row of zeros,

$$A(s) = 2s^4 + 8 \qquad \frac{dA(s)}{ds} = 8s^3$$

s^6	1	2	4	8	0
s^5	3	0	12	0	
s^4	2	0	8	0	
s^3	8	0	0		
s^2	$0 \approx \varepsilon$	8	0		
s^1	$-\frac{64}{\varepsilon}$	0			
s^0	8				

There are two sign changes, therefore two roots of the characteristic equation are located in the right half-plane and the system is unstable.

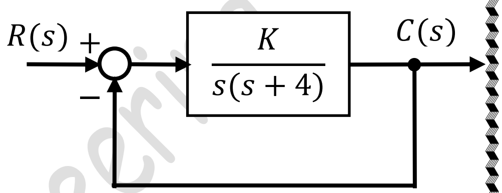
Chapter 7

Root Locus Method:

Transient response of a control system is governed by location of the roots of characteristic equation (zeros of characteristic function). Therefore factored form of characteristic equation plays an important role in transient response of control systems. Then transient behavior could be improved by selecting appropriate locations of roots or zeros. Therefore the trajectories (root loci) of the roots of characteristic equation should be investigated when a certain system parameter varies such as K.

Example 1:

Figure shows Laplace transform of a control system when all initial conditions are zero, determine the response $c(t)$ when $r(t) = 0$ and $c(0) = \dot{c}(0) = 1$, for each of the following cases:



$(a)\ K = 0,\quad (b)\ K = 3,\quad (c)\ K = 4,\quad (d)\ K = 8$

Then, plot the results of factored form on s-plane.

Solution:

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+4)}}{1+\frac{K}{s(s+4)}} = \frac{K}{s^2+4s+K}$$

Differential equation is:

$$\frac{c(t)}{r(t)} = \frac{K}{D^2+4D+K}$$

$$(D^2+4D+K)c(t) = Kr(t)$$

For $r(t) = 0$, Laplace transform,

$$s^2C(s) - sc(0) - \dot{c}(0) + 4[sC(s) - c(0)] + KC(s) = 0$$

$$C(s) = \frac{s+5}{s^2+4s+K}$$

a) For $K = 0$

$$C(s) = \frac{s+5}{s^2+4s+0} = \frac{s+5}{s(s+4)}$$

Techniques of Partial Fraction Expansion yields:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+4}$$

$$K_1 = \lim_{s\to 0}\left[s\frac{s+5}{s(s+4)}\right] = \frac{5}{4} \qquad K_2 = \lim_{s\to -4}\left[(s+4)\frac{s+5}{s(s+4)}\right] = -\frac{1}{4}$$

$$C(s) = \frac{5/4}{s} - \frac{1/4}{s+4}$$

Laplace inverse gives transient response or time solution,

$$c(t) = (5/4) - (1/4)e^{-4t}$$

For, $K = 0$, the factored form of characteristic equation is, $s^2+4s+0 = s(s+4)$

b) For $K = 3$

$$C(s) = \frac{s+5}{s^2+4s+3} = \frac{s+5}{(s+1)(s+3)}$$

Techniques of Partial Fraction Expansion yields:

$$C(s) = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

$$K_1 = \lim_{s \to -1}\left[(s+1)\frac{s+5}{(s+1)(s+3)}\right] = 2 \qquad K_2 = \lim_{s \to -3}\left[(s+3)\frac{s+5}{(s+1)(s+3)}\right] = -1$$

$$C(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

Laplace inverse gives the transient response or time solution,

$$c(t) = 2e^{-t} - e^{-3t}$$

For $K = 3$, factored form of characteristic equation is, $s^2 + 4s + 3 = (s+1)(s+3)$

c) For $K = 4$

$$C(s) = \frac{s+5}{s^2+4s+4} = \frac{s+5}{(s+2)^2}$$

Techniques of Partial Fraction Expansion yields:

$$C(s) = \frac{C_2}{(s+2)^2} + \frac{C_1}{s+2}$$

$C_2 = \lim_{s \to -2}\left[(s+2)^2 \frac{s+5}{(s+2)^2}\right] = 3$

$C(1) = \frac{1+5}{(1+2)^2} = \frac{3}{(1+2)^2} + \frac{C_1}{1+2}$ $\qquad C_1 = 1$

$$C(s) = \frac{3}{(s+2)^2} + \frac{1}{s+2}$$

Laplace inverse gives transient response or time solution,

$$c(t) = (1+3t)e^{-2t}$$

For $K = 4$, factored form of characteristic equation is,

$$s^2 + 4s + 4 = (s+2)^2$$

d) For $K = 8$

$$C(s) = \frac{s+5}{s^2+4s+8}$$

$$s^2 + 4s + 8 = s^2 - 2as + a^2 + b^2$$

$$r_1, r_2 = -2 \mp j2$$

$$C(s) = \frac{s+5}{[s-(-2+j2)][s-(-2-j2)]}$$

Transient response or time solution is;

$$c(t) = \frac{1}{b}|K(a+jb)|e^{at}\sin(bt+\alpha)$$

$$K(a+jb) = \left[(s^2+4s+8)\frac{s+5}{s^2+4s+8}\right]_{s=-2+j2} = 3 + j2$$

$$|K(a+jb)| = \sqrt{9+4} = \sqrt{13}$$

$$\alpha = tan^{-1}\frac{2}{3} = 33.6°$$

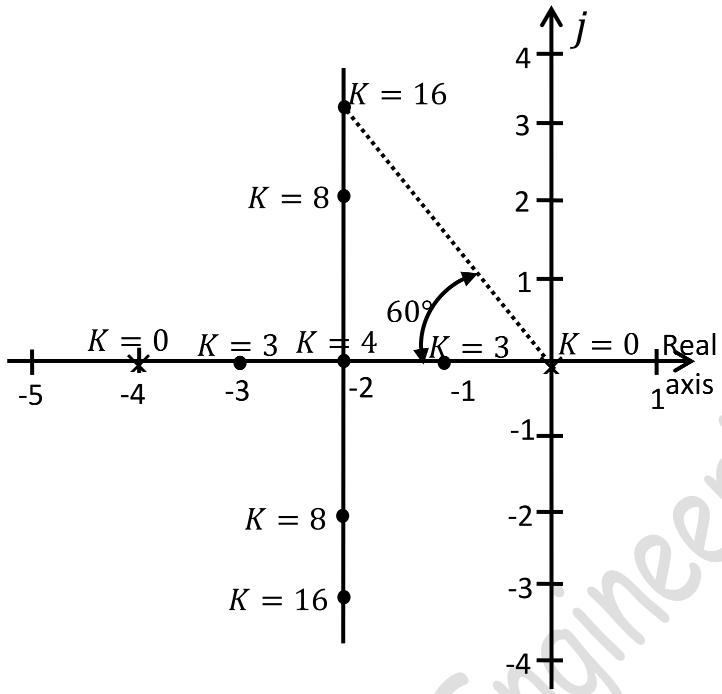
Then,

$$c(t) = \frac{\sqrt{13}}{2}e^{-2t}\sin(2t + 33.6°)$$

For $K = 8$, characteristic equation has complex conjugate roots located at $-2 \mp j2$ the factored form is,

$$s^2 + 4s + 8 = [s-(-2+j2)][s-(-2-j2)]$$

Finally, roots of the characteristic equation with different values of K could be plotted in Figure shown such a plot is called a root-locus plot.



Let it be desired to have a damping ratio $\xi = 0.5$, then, $\beta = \cos^{-1}\xi = 60°$, Line with angle $\beta = 60°$ intersects root-locus plot at corresponding roots (graphically):

$$r_{1,2} - 2 \mp j2\sqrt{3}$$

The value of $K = 16$ is determined as,

$$\left[s - \left(-2 + j2\sqrt{3}\right)\right]\left[s - \left(-2 - j2\sqrt{3}\right)\right] = s^2 + 4s + 16$$

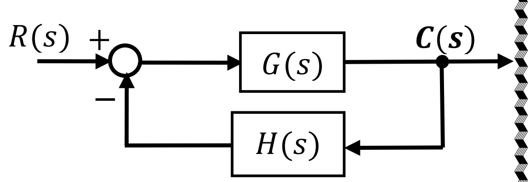
Thus, $\omega_n^2 = 16$ $\omega_n = 4$ and $2\xi\omega_n = 4$ $\xi = 0.5$

Root-Locus Method:

Root-locus Method is a graphical approach for examining how the roots of characteristic equation change with variation of a certain parameter such as gain K. Root locus is essentially the trajectory of roots as K varies from 0 to infinity. For each value of K the corresponding roots of the characteristic equation may be determined directly from the root-locus plot.

1. Factored Form of the Characteristic Equation:

Consider Laplace transformed block diagram:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$1 + G(s)H(s) = 0$ Characteristic Equation

$G(s)H(s)$ Open Loop Transfer Function

$$1 + \frac{N_G N_H}{D_G D_H} = 0$$

$N_G N_H$ is the numerator of the open loop transfer function,

$D_G D_H$ is the denominator of the open loop transfer function

Factorization yields,

$$N_G N_H = K(s - z_1)(s - z_2)\cdots\cdots(s - z_m)$$

$$D_G D_H = (s - p_1)(s - p_2)\cdots\cdots(s - p_n)$$

Where, the gain K is the static loop sensitivity which is the product of all the constant terms in the control loop. Then, the characteristic equation,

$$\frac{K(s-z_1)(s-z_2)\cdots\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots\cdots(s-p_n)}=-1$$

$(s-z_1)(s-z_2)\cdots\cdots(s-z_m)$, are called the zeros of open loop transfer function which are roots of $N_G N_H=0$

$(s-p_1)(s-p_2)\cdots\cdots(s-p_n)$, are called the poles of open loop transfer function which are roots of $D_G D_H=0$

General factored form of characteristic equation,

$$\frac{(s-p_1)(s-p_2)\cdots\cdots(s-p_n)}{(s-z_1)(s-z_2)\cdots\cdots(s-z_m)}=-K$$

In root-locus plot, poles of the characteristic equation $p_1,p_2,\cdots\cdots,p_n$ are plotted as $(\times)$ and zeros of the characteristic equation $z_1,z_2,\cdots\cdots,z_m$ are plotted as (O) where n is the number of poles and m is the number of zeros and usually, $n>m$.

- Poles of characteristic equation are obtained at $K=0$ then,
$$(s-p_1)(s-p_2)\cdots\cdots(s-p_n)=0$$
- Zeros of the characteristic equation are obtained at $K=\infty$, then,
$$(s-z_1)(s-z_2)\cdots\cdots(s-z_m)=0$$

Note: Since $K=0$ at poles, then a locus starts at each pole and terminates at a zero where $K=\infty$

➢ Magnitude condition for the factored form of characteristic equation,

$$\frac{|s-p_1||s-p_2|\cdots\cdots|s-p_n|}{|s-z_1||s-z_2|\cdots\cdots|s-z_m|}=|-K|$$

Magnitude condition is used to obtain values of the gain K at any point on root-locus plot.

➢ Angle condition for the factored form of characteristic equation,

$$[\measuredangle(s-p_1)+\measuredangle(s-p_2)+\cdots+\measuredangle(s-p_n)]-[\measuredangle(s-z_1)+\measuredangle(s-z_2)+\cdots+\measuredangle(s-z_m)]=\measuredangle(-K)$$

$$\sum_{i=1}^{n}\measuredangle(s-p_i)-\sum_{i=1}^{m}\measuredangle(s-z_i)=180^\circ\mp k360^\circ \qquad k=0,1,2,3,\cdots$$

In order that a point in s-plane be a root of characteristic equation and thus lies on a locus of the roots, it is necessary that this point satisfies angle condition.

2. **<u>Location of Loci Along Real Axis:</u>**

Loci may be investigated along real axis using angle condition,

$$\sum_{i=1}^{n}\measuredangle(s-p_i)-\sum_{i=1}^{m}\measuredangle(s-z_i)=180^\circ\mp k360^\circ \qquad k=0,1,2,3,\cdots$$

Complex conjugate poles or zeros do not affect the location of the loci on the real axis.

General Important Rule,

I. There is never a locus to the right of the first pole (O) or zero $(\times)$ on the real axis, but there is always a locus to the left of the first pole (O) or zero $(\times)$.

II. There is never a locus to the left of the second pole (O) or zero $(\times)$, but there is always a locus to the left of the third pole (O) or zero $(\times)$.

III. There is never a locus to the left of the fourth pole (O) or zero $(\times)$, but there is always a locus to the left of the fifth pole (O) or zero $(\times)$.

3. **Asymptotes as s Approaches Infinity:**

Since loci starts at pole $(\times)$ and terminates at zero (O), so when, $n \neq m$, asymptotes represent location of loci for large values of s

- $n-m$ Number of distinct asymptotes
- Asymptotes intersects real axis at point σ_c (point of intersection)

$$\sigma_c = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

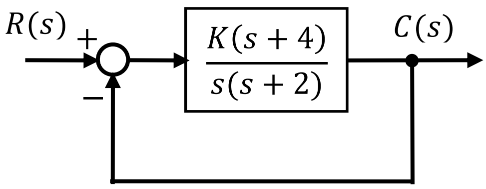
- Angles of asymptotes:

$$\measuredangle s = \frac{180^\circ \mp k360^\circ}{n-m} \qquad k = 0, 1, 2, 3, \cdots$$

4. **Break-away and Break-in Points:**
   - Break-away point is the point where a locus breaks away from real axis while Break-in point the locus breaks into real axis.
   - For specified segment on real axis gain K has its maximum value at break-away point while it has minimum value at break-in point.
   - These two points could be obtained by setting (dK/ds) is equal to zero.
5. **Angles of Departure and Arrival for Complex Conjugate Roots:**
   - Angle of departure is the angle at which a locus departs or leaves a complex pole $(\times)$ whereas angle of arrival is the angle at which a locus arrives or approaches a complex zero (o).
   - Their values are obtained by taking a trial point s closer to complex pole $(\times)$ or complex zero (o) and applying angle condition.
6. **Points Where Root Locus Cross the Imaginary Axis:**
   - Root locus plot may cross the imaginary axis.
   - Value of corresponding K at which root-locus plot crosses the imaginary axis could be determined using principles of Routh's Stability Criterion.
   - Complex conjugate roots which cross the imaginary axis are determined from the auxiliary equation which is existed in Routh's array.
7. **Some Notes Taken into Consideration to Construct Complete Root-locus plot:**
   - Root locus is symmetrical about real axis.
   - Determine a sufficient number of points that satisfy the angle condition.

Example 2:

Construct the root-locus plot for the system shown in Figure then examine locus on point (-8) and determine the values of K at which the system becomes unstable.



Solution:

1. **Factored Form of the Characteristic Equation**

$$\frac{C(s)}{R(s)} = \frac{K(s+4)}{s(s+2)+K(s+4)}$$

$$s(s+2)+K(s+4)=0 \qquad \textit{Characteristic equation}$$

Factored form of the characteristic equation:

$$\frac{s(s+2)}{(s+4)} = -K$$

Number of poles, $n=2$ $p_1 = 0$ $p_2=-2$ Number of zeros, $m=1$ $z_1=-4$

2. **Location of Loci Along Real Axis:**

$$\sum_{i=1}^{n} \measuredangle (s-p_i) - \sum_{i=1}^{m} \measuredangle (s-z_i) = 180^\circ \mp k360^\circ \qquad k=0,1,2,3,\cdots$$

$$[\measuredangle (s-0)+\measuredangle (s+2)]-\measuredangle (s+4) = [\phi_1+\phi_2]-\phi_3$$

- For right side of $p_1 = 0$, angle condition,
  $= [0^\circ + 0^\circ] - 0^\circ = 0^\circ \neq 180^\circ \mp k360^\circ$
  It doesn't satisfy angle condition, no locus on right side of $p_1 = 0$
- Between $p_1 = 0$ and $p_2 = -2$, angle condition,
  $= [180^\circ + 0^\circ] - 0^\circ = 180^\circ = 180^\circ \mp k360^\circ$
  Angle condition is satisfied, there is a locus between $p_1 = 0$ and $p_2 = -2$
- Between $p_2 = -2$ and $z_1 = -4$, angle condition,
  $= [180^\circ + 180^\circ] - 0^\circ = 360^\circ \neq 180^\circ \mp k360^\circ$
  Angle condition doesn't satisfy, there is no locus between $p_2 = -2$ and $z_1 = -4$
- For left side of $z_1 = -4$, the angle condition,
  $= [180^\circ + 180^\circ] - 180^\circ = 180^\circ = 180^\circ \mp k360^\circ$
  Angle condition is satisfied, there is a locus to left side of $z_1 = -4$

3. **Asymptotes as s Approaches Infinity:**

$n - m = 2 - 1 = 1$ Number of asymptotes

Intersection point of asymptotes with real axis,

$$\sigma_c = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{0+(-2)-(-4)}{1} = 2$$

Angles of asymptotes,

$$\measuredangle s = \frac{180^\circ \mp k360^\circ}{n-m} \qquad k=0,1,2,3,\cdots \qquad \measuredangle s = \frac{180^\circ}{1} = 180^\circ$$

4. **Break-away and Break-in Points:**

$$-K = \frac{s(s+2)}{s+4} \qquad -\frac{dK}{ds} = \frac{(s+4)(2s+2)-s(s+2)}{(s+4)^2} = 0$$

$(s+4)(2s+2) - s(s+2) = 0 \qquad s^2 + 8s + 8 = 0 \qquad s_1 = -1.171$ and $s_2 = -6.828$

For $\sigma_{b1} = -1.171$, $K = |-K| = \frac{|-1.171||-1.171+2|}{|-1.171+4|} = 0.343$ break- away point

For $\sigma_{b2} = -6.828$, $K = |-K| = \frac{|-6.828||-6.828+2|}{|-6.828+4|} = 11.656$ break-in point

5. **Angles of Departure and Arrival for Complex Conjugate Roots:**

No angles of departure and arrival because no complex roots in characteristic equation.

6. **Points Where Root Loci Cross the Imaginary Axis:**

$$s(s+2) + K(s+4) = 0 \qquad s^2 + (2+K)s + 4K = 0$$

Routh's Array

s^2	1	$4K$	0
s^1	$2+K$	0	
s^0	$4K$		

The value of K which makes a row of zero is, $K = -2$,

Then, the auxiliary equation,

$A(s) = s^2 + 4K = 0 \qquad s^2 - 8 = 0 \qquad s_{1,2} = \mp 2\sqrt{2}$ (real)

No points cross Imaginary axis

7. Some Additional points:

- $s = -2 + j\omega$ Determined from characteristic equation,

  $s(s+2) + K(s+4) = 0$ *Characteristic equation*

  $(-2 + j\omega)(-2 + j\omega + 2) + K(-2 + j\omega + 4) = 0$ $(-\omega^2 + 2K) + j(K-2)\omega = 0$

  For complex number to be zero, its real and imaginary parts should be zero,

  $$-\omega^2 + 2K = 0 \cdots\cdots\cdots\cdots (1)$$
  $$(K-2)\omega = 0 \cdots\cdots\cdots\cdots (2)$$

  Non-trivial solution yields, $K = 2$ and, $\omega = \mp 2$ then, $s = -2 \mp j2$

- $s = -6 + j\omega$

  $s(s+2) + K(s+4) = 0$ *Characteristic equation*

  $(-6 + j\omega)(-6 + j\omega + 2) + K(-6 + j\omega + 4) = 0$

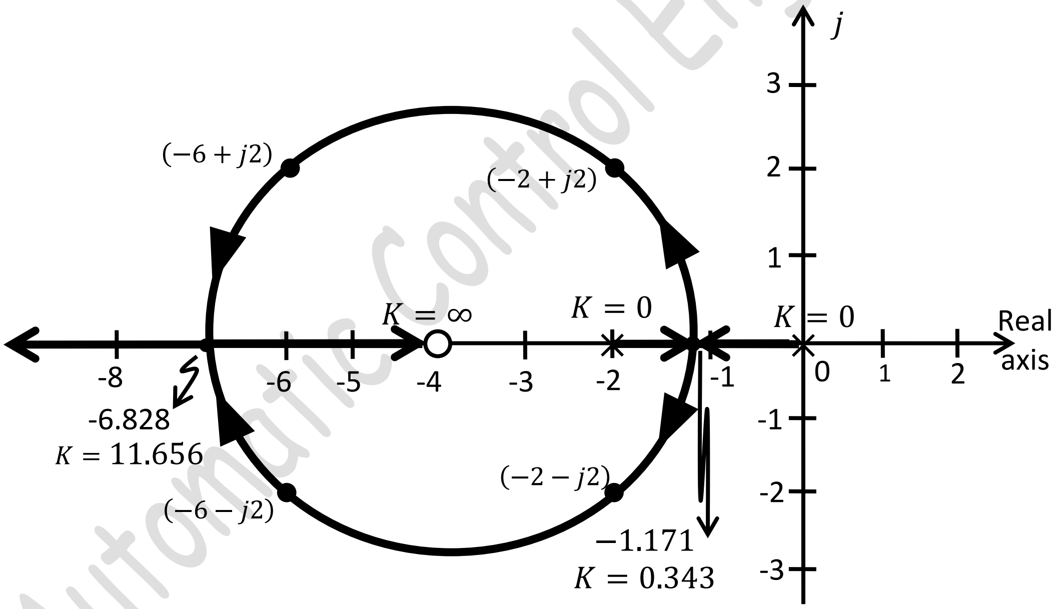
  $(-\omega^2 - 2K + 24) + j(K-10)\omega = 0$

  For complex number to be zero, its real and imaginary parts should be zero,

  $$-\omega^2 - 2K + 24 = 0 \cdots\cdots\cdots\cdots (1)$$
  $$(K-10)\omega = 0 \cdots\cdots\cdots\cdots (2)$$

  Non-trivial solution yields, $K = 10$ and, $\omega = \mp 2$ then, $s = -6 \mp j2$

Note: Locus follows circular path in the case of two poles and one zero.



Point (-8) is located to the left of the first zero $(z_1 = -4)$ where it is a locus. Or it could be examined by angle condition of the factored form of the characteristic equation,

$$= [180° + 180°] - 180° = 180° = 180° \mp k360°$$

Where, angle condition is satisfied.

Magnitude condition at point (-8)

$$K = |-K| = \frac{|-8||-8+2|}{|-8+4|} = 12$$

$s(s+2) + K(s+4) = 0$ Characteristic Equation

$$s^2 + 14s + 48 = 0$$

Solve yields, $s_1 = -8$ and $s_2 = -6$

Values of K at which the system becomes unstable is determined using Routh's Stability Criterion.

$s(s+2)+K(s+4)=0$ Characteristic Equation

$$s^2+(2+K)s+4K=0$$

Routh's Array,

s^2	1	$4K$	0
s^1	$2+K$	0	
s^0	$4K$	0	

For the system to be stable,

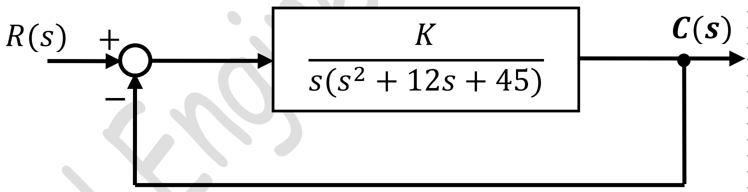
$2+K>0$ $\quad K>-2$

$4K>0$ $\quad K>0$

System becomes unstable for the values of K, $\quad K<0$

Example 3:

Sketch the root-locus plot for the closed loop control system shown in Figure.



Solution:

Transfer function is,

$$\frac{C(s)}{R(s)}=\frac{K}{s(s^2+12s+45)+K}$$

Characteristic equation,

$$s(s^2+12s+45)+K=0$$

1. **Factored form of the characteristic equation:**

$$s(s^2+12s+45)=-K$$

Factored form of characteristic equation,

$$s[s-(-6+j3)][s-(-6-j3)]=-K$$

- Number of poles, n = 3 (there are three loci)

$$p_1=0, \qquad p_2,p_3=-6\mp j3$$

- Number of zeros = m = 0

2. **Location of Loci Along Real Axis:**

$$\sum_{i=1}^{n}\sphericalangle(s-p_i)-\sum_{i=1}^{m}\sphericalangle(s-z_i)=180^\circ\mp k360^\circ \qquad k=0,1,2,3,\cdots$$

$$[\sphericalangle(s-p_1)+\sphericalangle(s-p_2)+\sphericalangle(s-p_3)]=180^\circ\mp k360^\circ$$

$$[\emptyset_1+\emptyset_2+\emptyset_3]=180^\circ\mp k360^\circ$$

Since complex conjugate poles do not affect the location of the loci on the real axis.

$$\emptyset_1=180^\circ\mp k360^\circ$$

- Consider right of $p_1=0$, the angle condition, $\quad 0^\circ=180^\circ\mp k360^\circ$
  It doesn't satisfy angle condition, no locus on right side of $p_1=0$
- Consider left of $p_1=0$, angle condition, $\quad 180^\circ=180^\circ\mp k360^\circ$
  Angle condition is satisfied, there is a locus to left of $p_1=0$

3. **Asymptotes as s Approaches Infinity:**

$n-m=3-0=3 \quad$ *Number of distinct asymptotes*

$$\sigma_c = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{[0 + (-6 + j3) + (-6 - j3)] - 0}{3 - 0} = -4$$

$$\measuredangle s = \frac{180^\circ \mp k360^\circ}{n-m} \qquad k = 0, 1, 2, 3, \cdots$$

$$\measuredangle s = \frac{180^\circ \mp k360^\circ}{3} = 60^\circ \mp k120^\circ$$

- for $k = 0$ $\measuredangle s = 60^\circ \mp k120^\circ = 60^\circ$ $\emptyset_1 = 60^\circ$
- for $k = 1$ $\measuredangle s = 60^\circ \mp k120^\circ = 60^\circ \mp 120^\circ$ $\emptyset_2 \ and \ \emptyset_3 = 180^\circ \ or \ -60^\circ$

4. <u>Break-away and Break-in Points with Real Axis:</u>

$s(s^2 + 12s + 45) = -K$ $s^3 + 12s^2 + 45s = -K$

$-\frac{dK}{ds} = 3s^2 + 24s + 45 = 0$ or $s^2 + 8s + 15 = 0$ Solve yields, $s_1 = -3$ and $s_2 = -5$

- For $\sigma_{b1} = -3$, $K = |-K| = |(-3)[(-3)^2 + 12(-3) + 45]| = 54$ break- away point
- For $\sigma_{b2} = -5$, $K = |-K| = |(-5)[(-5)^2 + 12(-5) + 45]| = 50$ break-in point

5. <u>Angles of Departure and Arrival for Complex Conjugate Roots:</u>

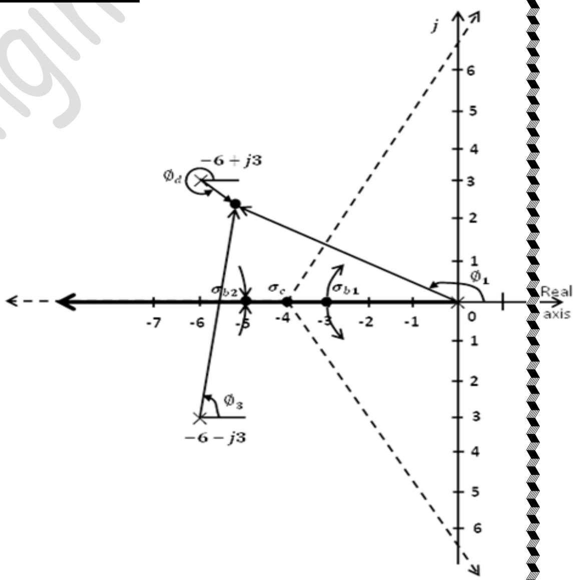
For the pole $-6 + j3$, angle condition,

$$\sum_{i=1}^{n} \measuredangle (s - p_i) - \sum_{i=1}^{m} \measuredangle (s - z_i) = 180^\circ \mp k360^\circ$$

$[\measuredangle (s - p_1) + \measuredangle (s - p_2) + \measuredangle (s - p_3)] = 180^\circ \mp k360^\circ$

$[\emptyset_1 + \emptyset_d + \emptyset_3] = 180^\circ \mp k360^\circ$

As trial point approaches pole $-6 + j3$, angles $\emptyset_1$ and $\emptyset_3$ could be easily obtained:



$\emptyset_1 = 180^\circ - \tan^{-1}\frac{3}{6} = 180^\circ - 26.56 = 153.434^\circ$, $\emptyset_3 = 90^\circ$

$[153.434^\circ + \emptyset_d + 90^\circ] = 180^\circ \mp k360^\circ$

$$\emptyset_d + 243.434^\circ = 180^\circ \mp k360^\circ$$

For $k = 0$, $\emptyset_d = -63.434^\circ$

For $k = 1$ $\emptyset_d = 296.566^\circ$ or -63.434°

Angle of departure from the pole $(-6 - j3)$, $\emptyset_d = 63.434^\circ$, or -296.566°

6. <u>Points Where Root Loci Cross the Imaginary Axis:</u>

$s(s^2 + 12s + 45) + K = 0$ $s^3 + 12s^2 + 45s + K = 0$

s^3	1	45	0
s^2	12	K	0
s^1	$\frac{540 - K}{12}$	0	
s^0	K		

$\frac{540 - K}{12} > 0$ $K < 540$

$K = 540$ $A(s) = 12s^2 + K = 0$ $12s^2 + 540 = 0$

$$s_{1,2} = \mp j3\sqrt{5} = \mp j6.7$$

7. <u>Some Additional points:</u>

$s(s^2 + 12s + 45) + K = 0$ Characteristic Equation

- Let's determine the Point $s = -2 + j\omega$

$$(-2 + j\omega)[(-2 + j\omega)^2 + 12(-2 + j\omega) + 45] + K = 0$$

$$(-6\omega^2 + K - 50) + j(9\omega - \omega^3) = 0$$

For complex number to be zero, its real and imaginary parts should be zero,

$$-6\omega^2 + K - 50 = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

$$9\omega - \omega^3 = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

$9\omega - \omega^3 = 0,\quad \omega(9 - \omega^2) = 0,\quad \omega(3 - \omega)(3 + \omega) = 0,\quad \omega = \mp 3 \qquad s_{1,2} = -2 \mp j3$

- Now determine the Point $s = -5.5 + j\omega$

$$(-5.5 + j\omega)[(-5.5 + j\omega)^2 + 12(-5.5 + j\omega) + 45] + K = 0$$

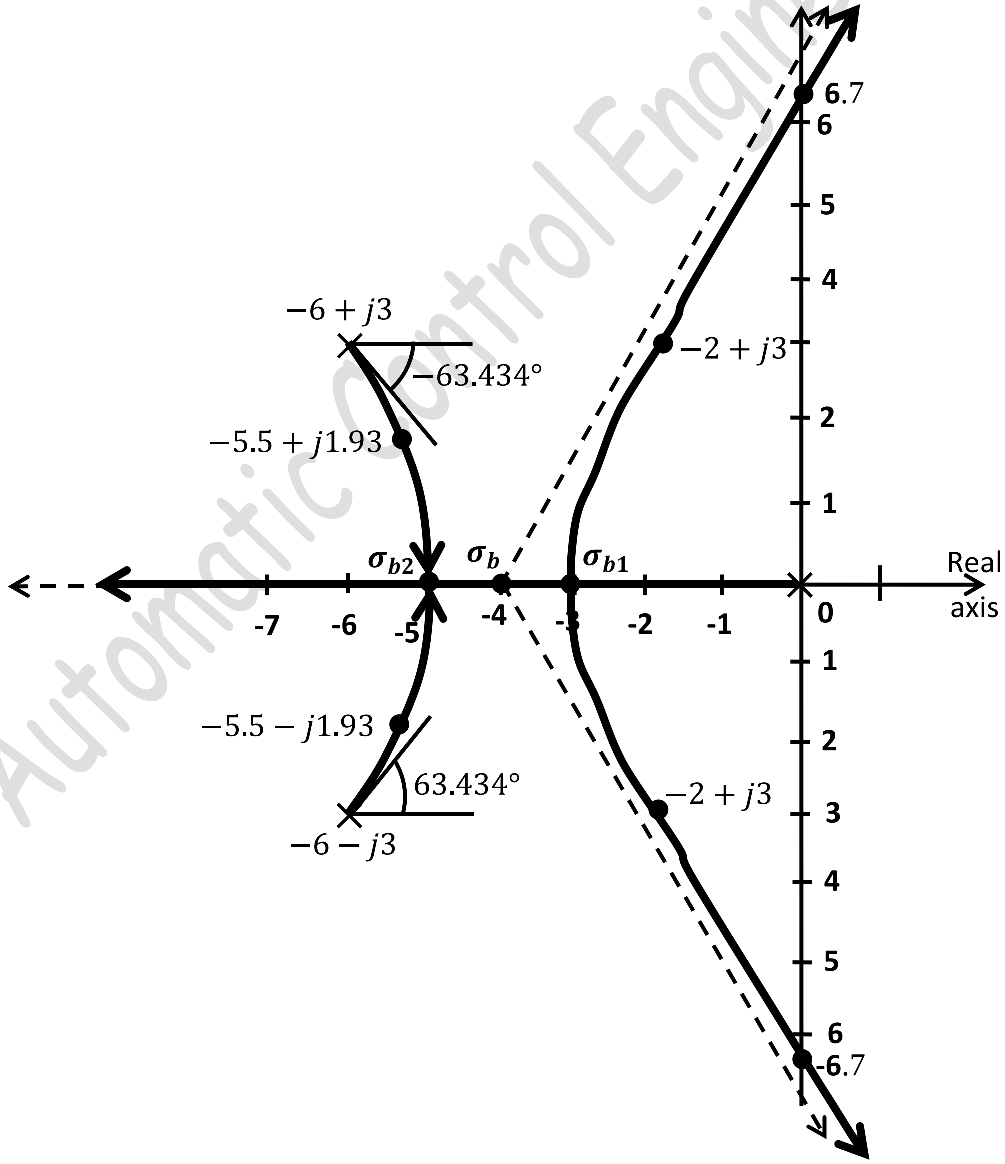
$$(-50.875 + 4.5\omega^2 + K) + j(3.75\omega - \omega^3) = 0$$

For complex number to be zero, its real and imaginary parts should be zero,

$$4.5\omega^2 + K - 50.875 = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

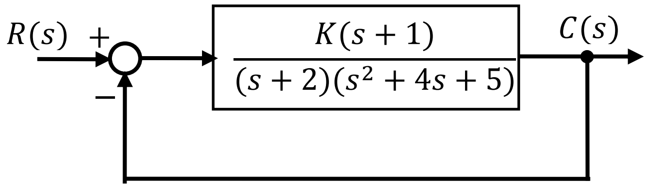
$$3.75\omega - \omega^3 = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

$\omega(3.75 - \omega^2) = 0,\quad \omega = 0,\quad 3.75 - \omega^2 = 0\quad \omega = \mp 1.93,\quad s_{1,2} = -5.5 \mp j1.93$



Example 4:

Sketch root-locus plot for the closed loop control system shown in Figure and determine values of K at which the system becomes unstable.



Solution: Transfer function is,

$$\frac{C(s)}{R(s)}=\frac{K(s+1)}{(s+2)(s^2+4s+5)+K(s+1)}$$

Characteristic equation,

$$(s+2)(s^2+4s+5)+K(s+1)=0$$

1. **Factored form of the characteristic equation:**

$$\frac{(s+2)(s^2+4s+5)}{s+1}=\frac{(s+2)[s-(-2+j)][s-(-2-j)]}{s+1}=-K$$

- Number of poles = n = 3 (there are three loci)
  $p_1=-2,\qquad p_2,p_3=-2\mp j$
- Number of zeros = m = 1 $\qquad z_1=-1$

2. **Investigate Location of Loci Along Real Axis:**

Using angle condition,

$$\sum_{i=1}^{n}\measuredangle(s-p_i)-\sum_{i=1}^{m}\measuredangle(s-z_i)=180°\mp k360° \qquad k=0,1,2,3,\cdots$$

$$\measuredangle(s+2)-\measuredangle(s+1)=\phi_1-\phi_2$$

- Right side of $z_1=-1$, angle condition, $\quad 0-0=0\neq 180°\mp k360°$
  It doesn't satisfy angle condition, no locus on right side of $z_1=-1$
- Between $z_1=-1$ and $p_1=-2$, angle condition,
  $$0-180=-180=180°\mp k360°$$
  Angle condition is satisfied, there is a locus between $z_1=-1$ and $p_1=-2$
- Left side of $p_1=-2$ angle condition,
  $$180-180=0\neq 180°\mp k360°$$
  It doesn't satisfy angle condition, no locus to left of $p_1=-2$

3. **Asymptotes as s Approaches Infinity:**

$n-m=3-1=2$ Number of distinct asymptotes

$$\sigma_c=\frac{\sum_{i=1}^{n}p_i-\sum_{i=1}^{m}z_i}{n-m}=\frac{[-2-2+j-2-j]-[-1]}{3-1}=-\frac{5}{2}$$

$$\measuredangle s=\frac{180°\mp k360°}{n-m}\qquad k=0,1,2,3,\cdots \qquad\qquad \measuredangle s=\frac{180°\mp k360°}{2}=90°\mp k180°$$

- $for\ k=0,\ \measuredangle s=90°\mp k180°=90°,\qquad \emptyset_1=90°$
- $for\ k=1,\ \measuredangle s=90°\mp k180°=90°\mp 180°,\qquad \emptyset_2=-90°\ or\ 270°$

4. **Break-away and Break-in Points with Real Axis:**

$$-K=\frac{(s+2)(s^2+4s+5)}{s+1}$$

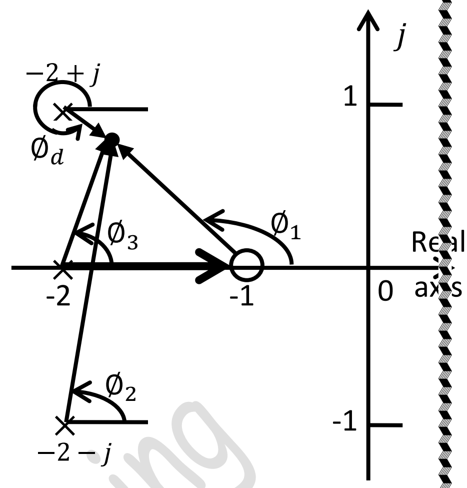
$$-\frac{dK}{ds}=\frac{(s+1)[(s+2)(2s+4)+(s^2+4s+5)]-[(s+2)(s^2+4s+5)]}{(s+1)^2}=0$$

$$s^3+3s^2+4s+1=0 \qquad s_1,s_2=-1.3412\mp j1.1615 \qquad s_3=-0.3177$$

It is well noticed that no any break-away and break-in points on real axis.

5. Angles of Departure and Arrival for Complex Conjugate Roots:

Value of angle of departure is obtained by taking a trial point s closer to the pole $-2+j$ and applying angle condition,



$$\sum_{i=1}^{n} \measuredangle (s-p_i) - \sum_{i=1}^{m} \measuredangle (s-z_i) = 180^\circ \mp k360^\circ \quad k=0,1,2,3,\cdots$$

$$[\measuredangle (s-p_1) + \measuredangle (s-p_2) + \measuredangle (s-p_3)] - [\measuredangle (s-z_1)] = 180^\circ \mp k360^\circ$$

$$[\emptyset_d + \emptyset_2 + \emptyset_3] - [\emptyset_1] = 180^\circ \mp k360^\circ$$

As trial point approaches pole $-2+j$, angles $\emptyset_1, \emptyset_2$, and $\emptyset_3$ could be easily obtained:

$$180^\circ - \emptyset_1 = \tan^{-1}\frac{1}{1} = 45^\circ, \qquad \emptyset_1 = 135^\circ \qquad \emptyset_2 = \emptyset_3 = 90^\circ$$

$$[\emptyset_d + 90^\circ + 90^\circ] - [135^\circ] = 180^\circ \mp k360^\circ$$

$$\emptyset_d + 45^\circ = 180^\circ \mp k360^\circ$$

For $k=0$, $\emptyset_d = 135^\circ$,

For $k=1$, $\emptyset_d = -225^\circ$ $\qquad \emptyset_d = 135^\circ$, or -225°

Angle of departure from pole $(-2-j)$ is $\emptyset_d = -135^\circ$, or 225°

6. Points Where Root Loci Cross the Imaginary Axis:

$$(s+2)(s^2+4s+5) + K(s+1) = 0 \qquad \text{Characteristic Equation}$$

$$s^3 + 6s^2 + (K+13)s + (10+K) = 0$$

s^3	1	$K+13$	0
s^2	6	$10+K$	0
s^1	$\frac{5K+68}{6}$	0	
s^0	$10+K$		

$$\frac{5K+68}{6} > 0 \qquad K > -13.6 \qquad 10+K > 0 \qquad K > -10$$

$K = -13.6$ $\qquad A(s) = 6s^2 + 10 + K = 0 \qquad 6s^2 - 3.6 = 0 \qquad s_{1,2} = \mp\sqrt{0.6}$ (real)

No points where root loci cross Imaginary axis,

For system to be stable K should be,

$$K > -13.6$$

System becomes unstable for values of K, $\qquad K < -13.6$

7. Some Additional points:

Point $s = -2.2 + j\omega$ could be determined,

$$(-2.2 + j\omega + 2)[(-2.2+j\omega)^2 + 4(-2.2+j\omega) + 5] + K(-2.2 + j\omega + 1) = 0$$

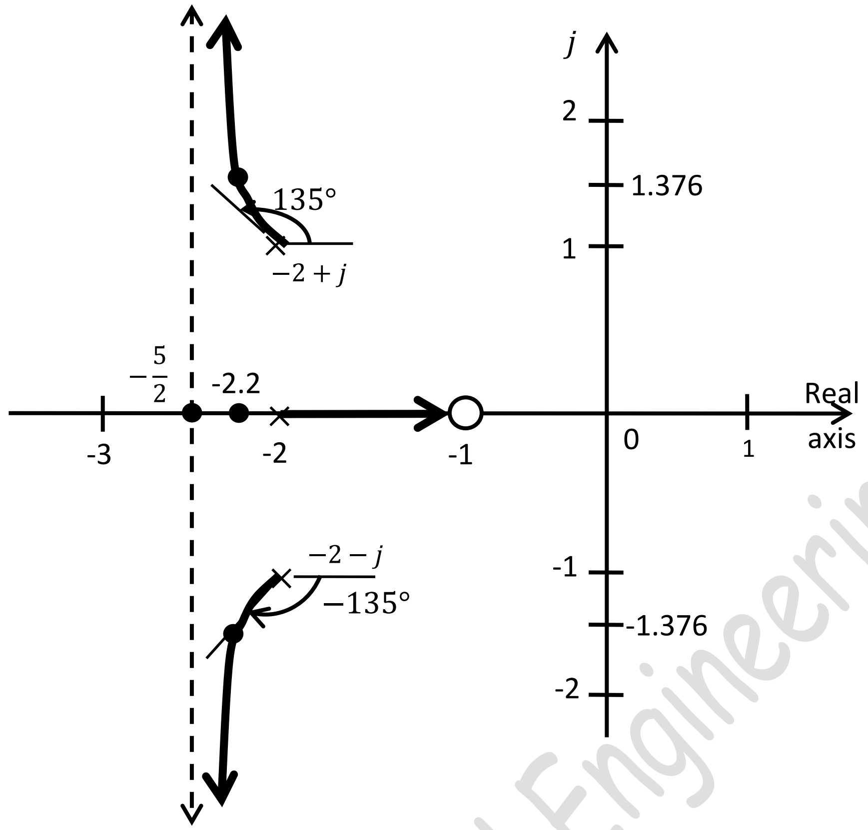
$$-0.208 + 0.08j\omega + 0.2\omega^2 + 1.04j\omega + 0.4\omega^2 - j\omega^3 - 1.2K + j\omega K = 0$$

For complex number to be zero, its real and imaginary parts should be zero,

$$\omega^2 - 0.346 - 2K = 0 \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

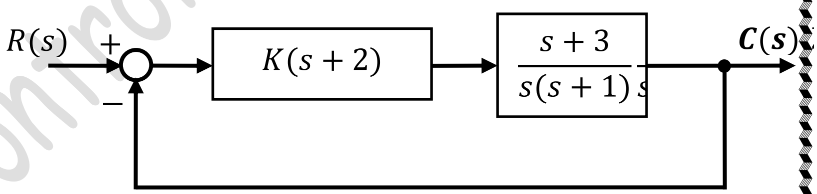
$$-\omega^2 + 1.12 + K = 0 \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Solve these equations simultaneously: $\omega = \mp 1.376$ and $s = -2.2 \mp j1.376$



Example 5:

Consider the closed-loop control system shown and plot its poles and zeros.



Solution: Transfer function

$$\frac{C(s)}{R(s)} = \frac{K(s+2)(s+3)}{s(s+1)+K(s+2)(s+3)}$$

$s(s+1)+K(s+2)(s+3)=0$ Characteristic equation,

1. **Factored form of the characteristic equation:**

Factored form of the characteristic equation:

$$\frac{s(s+1)}{(s+2)(s+3)} = -K$$

$n=2,\quad p_1=0,\ p_2=-1,$ and, $m=2,\quad z_1=-2,\quad z_2=-3$

2. **Investigate Location of Loci Along Real Axis:**

$$\sum_{i=1}^{n} \measuredangle (s-p_i) - \sum_{i=1}^{m} \measuredangle (s-z_i) = 180^\circ \mp k360^\circ \qquad k=0,1,2,3,\cdots$$

$$[\measuredangle (s-0)+\measuredangle (s+1)] - [\measuredangle (s+2)+\measuredangle (s+3)] = 180^\circ \mp k360^\circ$$

$$[\phi_1+\phi_2]-[\phi_3+\phi_4] = 180^\circ \mp k360^\circ$$

- For right side of $p_1=0$, angle condition,
  $$= [0^\circ+0^\circ]-[0^\circ+0^\circ] = 0^\circ \neq 180^\circ \mp k360^\circ$$
  It doesn't satisfy angle condition, no locus on right side of $p_1=0$
- Between $p_1=0$ and $p_2=-1$,
  $= [180^\circ+0^\circ]-[0^\circ+0^\circ] = 180^\circ = 180^\circ \mp k360^\circ$
  Angle condition is satisfied, locus between $p_1=0$ and $p_2=-1$

- Between $p_2 = -1$ and $z_1 = -2$,
$$= [180° + 180°] - [0° + 0°] = 360° \neq 180° \mp k360°$$
Angle condition doesn't satisfy, no locus between $p_2 = -1$ and $z_1 = -2$
- Between $z_1 = -2$ and $z_1 = -3$
$$= [180° + 180°] - [180° + 0°] = 180° = 180° \mp k360°$$
Angle condition is satisfied, locus between $z_1 = -2$ and $z_1 = -3$
- For left side of $z_1 = -3$,
$$= [180° + 180°] - [180° + 180°] = 0° \neq 180° \mp k360°$$

It doesn't satisfy angle condition, no locus on left side of $z_1 = -3$

3. **Asymptotes as s Approaches Infinity:**

Number of asymptotes,
$$n - m = 2 - 2 = 0$$
No asymptotes will be existed

4. **Break-away and Break-in Points:**

$$\frac{s(s+1)}{(s+2)(s+3)} = -K \qquad -\frac{dK}{ds} = \frac{(s+2)(s+3)(2s+1) - s(s+1)(2s+5)}{[(s+2)(s+3)]^2} = 0$$

$$(s+2)(s+3)(2s+1) - s(s+1)(2s+5) = 0 \qquad 4s^2 + 12s + 6 = 0$$

$$s_1 = -0.634 \quad \text{and} \quad s_2 = -2.366$$

- For $s_1 = -0.634$, $K = |-K| = \frac{|-0.634||-0.634+1|}{|-0.634+2||-0.634+3|} = 0.0718$ break-away point
- For $s_2 = -2.366$, $K = |-K| = \frac{|-2.366||-2.366+1|}{|-2.366+2||-2.366+3|} = 14$ break-in point

5. **Angles of Departure and Arrival for Complex Conjugate Roots:**

No angles of departure and arrival because no complex roots in characteristic equation.

6. **Points Where Root Loci Cross the Imaginary Axis:**

$s(s+1) + K(s+2)(s+3) = 0$ Characteristic equation,
$$s^2 + \frac{1+5K}{1+K}s + \frac{6K}{1+K} = 0$$

Routh's Array

s^2	1	$\frac{6K}{1+K}$	0
s^1	$\frac{1+5K}{1+K}$	0	
s^0	$\frac{6K}{1+K}$		

$\frac{1+5K}{1+K} > 0$ $K > -\frac{1}{5}$ and $\frac{6K}{1+K} > 0$ $K > 0$

$K = -\frac{1}{5}$ should be substituted in the auxiliary equation,

$A(s) = s^2 + \frac{6K}{1+K} = 0$ $s^2 - \frac{3}{2} = 0$ $s_{1,2} = \mp\sqrt{\frac{3}{2}}$ (real)

No points cross Imaginary axis

7. **Some Additional points:**

- $s = -\frac{3}{2} + j\omega$ Determined from characteristic equation,
$$s(s+1) + K(s+2)(s+3) = 0 \qquad \text{Characteristic equation}$$
$$\left(-\frac{3}{2} + j\omega\right)\left(-\frac{3}{2} + j\omega + 1\right) + K\left(-\frac{3}{2} + j\omega + 2\right)\left(-\frac{3}{2} + j\omega + 3\right) = 0$$

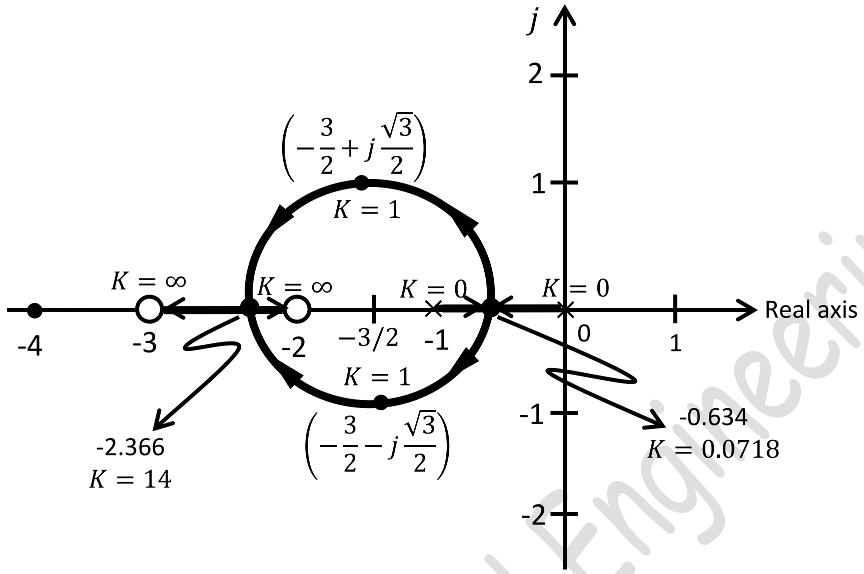
$$(1+K)\omega^2 - 0.75(1+K) + j2(1-K)\omega = 0$$

For complex number to be zero, its real and imaginary parts should be zero,

$$(1+K)\omega^2 - 0.75(1+K) = 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

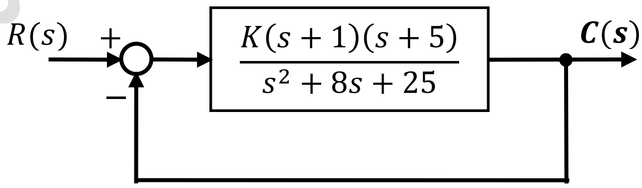
$$2(1-K)\omega = 0 \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Trivial solution must be discarded, $\omega = \mp\frac{\sqrt{3}}{2}$ and $s = -\frac{3}{2} \mp j\frac{\sqrt{3}}{2}$



Example 6:

Sketch root-locus plot for the closed loop control system shown in Figure then examine locus on point (-4) and determine the values of K at which the system becomes unstable.



Solution: Transfer function

$$\frac{C(s)}{R(s)} = \frac{K(s+1)(s+5)}{(s^2+8s+25)+K(s+1)(s+5)}$$

$$(s^2+8s+25)+K(s+1)(s+5) = 0$$ Characteristic equation,

1. **Factored form of the characteristic equation:**

$$\frac{(s^2+8s+25)}{(s+1)(s+5)} = -K \qquad \frac{[s-(-4+j3)][s-(-4-j3)]}{(s+1)(s+5)} = -K$$

$$n = 2, \quad p_1 = -4+j3, \; p_2 = -4-j3,$$
$$m = 2, \quad z_1 = -1, \quad z_2 = -5$$

2. **Investigate Location of Loci Along Real Axis:**

$$\sum_{i=1}^{n} \measuredangle (s-p_i) - \sum_{i=1}^{m} \measuredangle (s-z_i) = 180^\circ \mp k360^\circ \qquad k = 0,1,2,3,\cdots$$

$$[\measuredangle (s-p_1) + \measuredangle (s-p_1)] - [\measuredangle (s-z_1) + \measuredangle (s-z_2)] = 180^\circ \mp k360^\circ$$

$$[\phi_1+\phi_2] - [\phi_3+\phi_4] = 180^\circ \mp k360^\circ \qquad [\phi_1+\phi_2] = 180^\circ$$

$$-[\phi_3+\phi_4] = 180^\circ \mp k360^\circ$$

- For right side of $z_1 = -1$, angle condition,

$$= -[0^\circ + 0^\circ] = 0^\circ \neq 180^\circ \mp k360^\circ$$

It doesn't satisfy angle condition, no locus to right side of $z_1 = -1$

- Between $z_1 = -1$ and $z_2 = -5$,
  $= -[180° + 0°] = -180° = 180° \mp k360°$
  Angle condition is satisfied, locus between $z_1 = -1$ and $z_2 = -5$
- For left side of $z_2 = -5$,
  $$= -[180° + 180°] = 0° \neq 180° \mp k360°$$

It doesn't satisfy angle condition, no locus on left side of $z_2 = -5$

3. **Asymptotes as s Approaches Infinity:**

Number of asymptotes,

$$n - m = 2 - 2 = 0$$

No asymptotes will be existed

4. **Break-away and Break-in Points:**

$$\frac{s^2 + 8s + 25}{(s+1)(s+5)} = \frac{s^2 + 8s + 25}{s^2 + 6s + 5} = -K$$

$$-\frac{dK}{ds} = \frac{(s^2 + 6s + 5)(2s + 8) - (s^2 + 8s + 25)(2s + 6)}{(s^2 + 6s + 5)^2} = 0$$

$(s^2 + 6s + 5)(2s + 8) - (s^2 + 8s + 25)(2s + 6) = 0$ $\qquad s^2 + 20s + 55 = 0$

$s_1 = -3.29$ (break-in point) $\qquad s_2 = -16.7$ (negligible, no locus)

For $s_1 = -3.29$,

$$K = |-K| = \frac{|s^2 + 8s + 25|}{|s^2 + 6s + 5|} = \frac{|(-3.29)^2 + 8(-3.29) + 25|}{|(-3.29)^2 + 6(-3.29) + 5|} = 2.427$$

5. **Angles of Departure and Arrival for Complex Conjugate Roots:**

For the pole $-4 + j3$, angle condition,

$$\sum_{i=1}^{n} \measuredangle (s - p_i) - \sum_{i=1}^{m} \measuredangle (s - z_i) = 180° \mp k360°$$

$$[\measuredangle (s - p_1) + \measuredangle (s - p_2)] - [\measuredangle (s - z_1) + \measuredangle (s - z_2)] = 180° \mp k360°$$

$$[\emptyset_d + \emptyset_1] - [\emptyset_2 + \emptyset_3] = 180° \mp k360°$$

As trial point approaches pole $-6 + j3$, angles could be easily obtained:

$$\emptyset_1 = 90°$$

$$\emptyset_2 = 180° - \tan^{-1}\frac{3}{3} = 180° - 45° = 135°$$

$$\emptyset_3 = \tan^{-1}\frac{3}{1} = 71.565°$$

$$[\emptyset_d + 90°] - [135° + 71.565°] = 180° \mp k360°$$

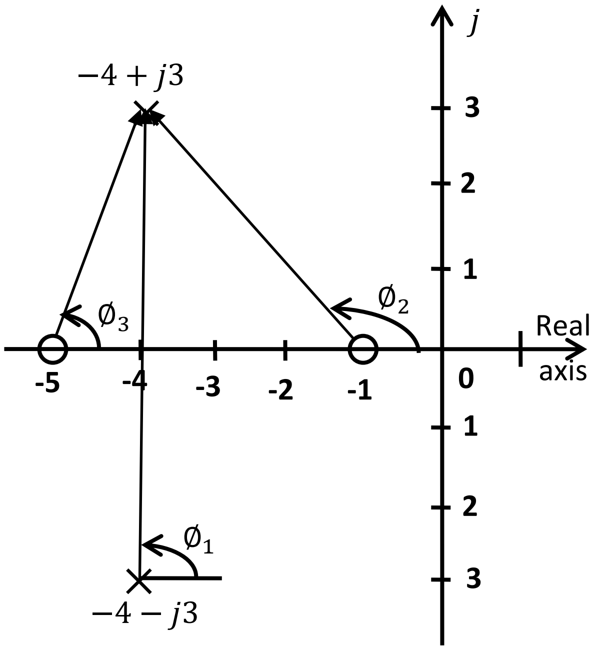
$$\emptyset_d - 116.565° = 180° \mp k360°$$

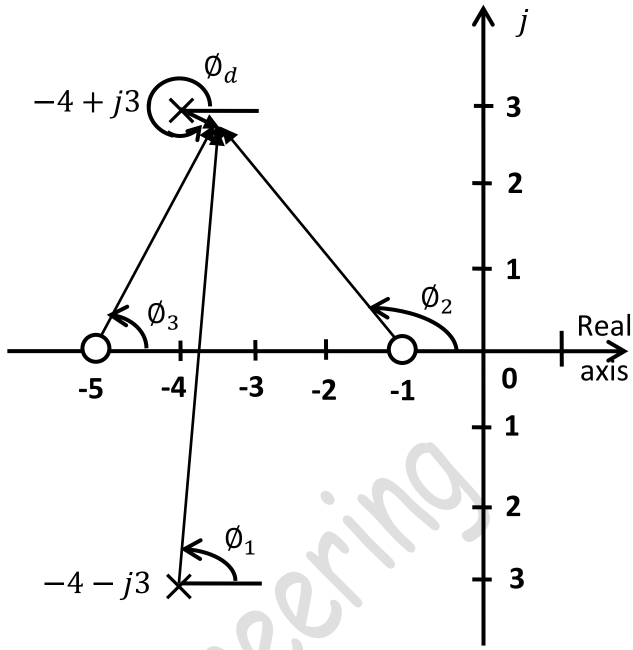
$$\emptyset_d = 296.565° \mp k360°$$

For $k = 0$, $\qquad \emptyset_d = 296.565°$ $\qquad$ Angle of Departure

For $k = 1$ $\qquad \emptyset_d = -63.435°$

Angle of departure from the pole $(-6 - j3)$ $\qquad \emptyset_d = 63.434°$, or $-296.566°$





6. Points Where Root Loci Cross the Imaginary Axis:

$(s^2+8s+25)+K(s+1)(s+5)=0$ Characteristic equation

$(1+K)s^2+(8+6K)s+5K=0$

Routh's Array

s^2	$1+K$	$5K$	0
s^1	$8+6K$	0	
s^0	$5K$		

For stability,

$1+K>0$ $\quad K>-1$

$8+6K>0$ $\quad K>-\frac{4}{3}$

$5K>0$ $\quad K>0$

$K=-\frac{4}{3}$ neglected, No points cross Imaginary axis

7. Some Additional points:

- $s=-3.5+j\omega$ Determined from characteristic equation,

$s^2+8s+25+K(s+1)(s+5)=0$ Characteristic equation

$(-3.5+j\omega)^2+8(-3.5+j\omega)+25+K(-3.5+j\omega+1)(-3.5+j\omega+5)=0$

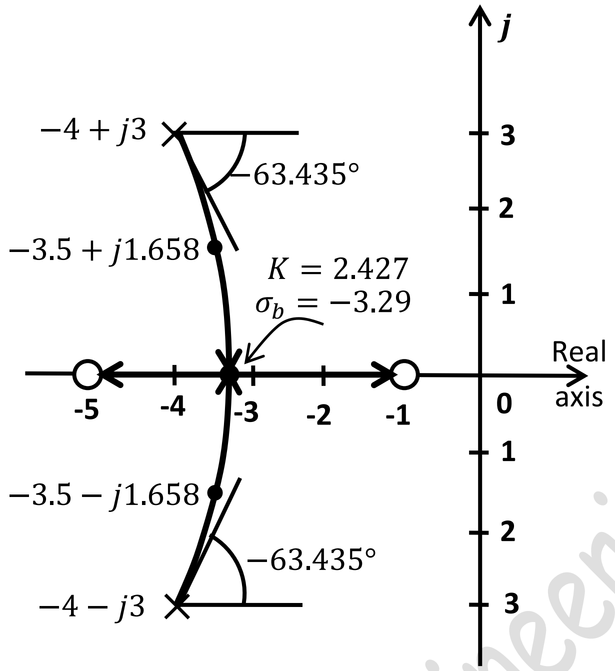
$-(1+K)\omega^2-3.75K+9.25+j(1-K)\omega=0$

For complex number to be zero, its real and imaginary parts should be zero,

$-(1+K)\omega^2-3.75K+9.25=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$

$(1-K)\omega=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$

Trivial solution must be discarded, $\omega=\mp1.658$ and $s=-3.5\mp j1.658$



Magnitude condition at point (-4)

$$K=|-K|=\frac{|s^2+8s+25|}{|(s+1)||(s+5)|}$$

$$K=|-K|=\frac{|(-4)^2+8(-4)+25|}{|(-4+1)||(-4+5)|}=3$$

$$s^2+8s+25+K(s+1)(s+5)=0 \qquad \text{Characteristic Equation}$$

$$s^2+8s+25+3(s+1)(s+5)=0$$

$$4s^2+26s+40=0$$

Solve yields, $s_1=-4$ and $s_2=-2.5$

Values of K at which the system becomes unstable is determined using Routh's Stability Criterion.

$$(s^2+8s+25)+K(s+1)(s+5)=0 \qquad \text{Characteristic equation}$$

$$(1+K)s^2+(8+6K)s+5K=0$$

Routh's Array

s^2	$1+K$	$5K$	0
s^1	$8+6K$	0	
s^0	$5K$		

For stability,

$1+K>0$ $\qquad K>-1$

$8+6K>0$ $\qquad K>-\frac{4}{3}$

$5K>0$ $\qquad K>0$

For the system to be stable, $K>-\frac{4}{3}$

For the system to be unstable, $K<-\frac{4}{3}$