Real Numbers System and Inqualities

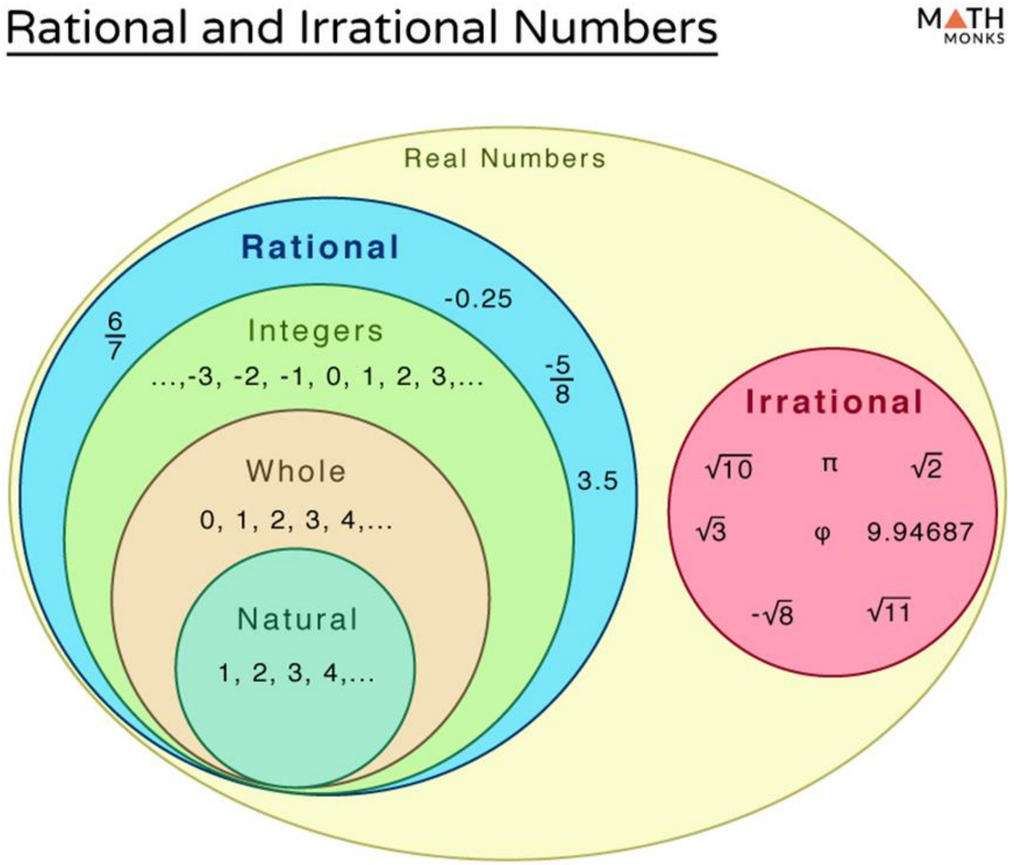
AUDAY K.A.

The Real numbers System and the Real Line

Much of calculus is based on properties of real number system. The real numbers consist of all the rational and irrational numbers.

The real number system has many subsets:

- Natural numbers (N) مجموعة الأعداد الطبيعية
- Whole numbers (W) مجموعة الأعداد الكلية
- Integers (Z) مجموعة الأعداد الصحيحة
- Rational numbers (Q) مجموعة الأعداد الكسرية او النسبية
- Irrational numbers (I) مجموعة الأعداد غير النسبية



- ***Natural numbers*** are the set of counting numbers, {1, 2, 3,…}.
- ***Whole numbers*** are the set of numbers that include 0 plus the set of natural numbers, {0, 1, 2, 3, 4, 5,…}.
- ***Integers*** are the set of whole numbers and their opposites, {…,-3, -2, -1, 0, 1, 2, 3,…}.
- ***Rational numbers*** are any numbers that can be expressed in the form of $\frac{a}{b}$, where *a* and *b* are integers, and b ≠ 0. They can always be expressed by using terminating decimals or repeating decimals.

Terminating decimals are decimals that contain a finite number of digits.

Examples:

- 36.8
- 0.125
- 4.5

$2.\overline{3} = 2.333333333333333333333333\ldots$ Period = 1

$0.\overline{456} = 0.456456456456456456456456\ldots$ Period = 3

$14.99\overline{34} = 14.99343434343434343434\ldots$ Period = 2

$12.\overline{714285} = 2.3 = 12.714285714285714285\ldots$ Period = 6

$0.1234\overline{5} = 0.123455555555555555555\ldots$ Period = 1

Repeating decimals are decimals that contain an infinite number of digits.

Venn Diagram

Venn diagram—a diagram consisting of circles or squares to show relationships of a set of data.

Example:

Classify all the following numbers as **natural**, **whole**, **integer**, **rational**, or **irrational**. List all that apply.

a. 117

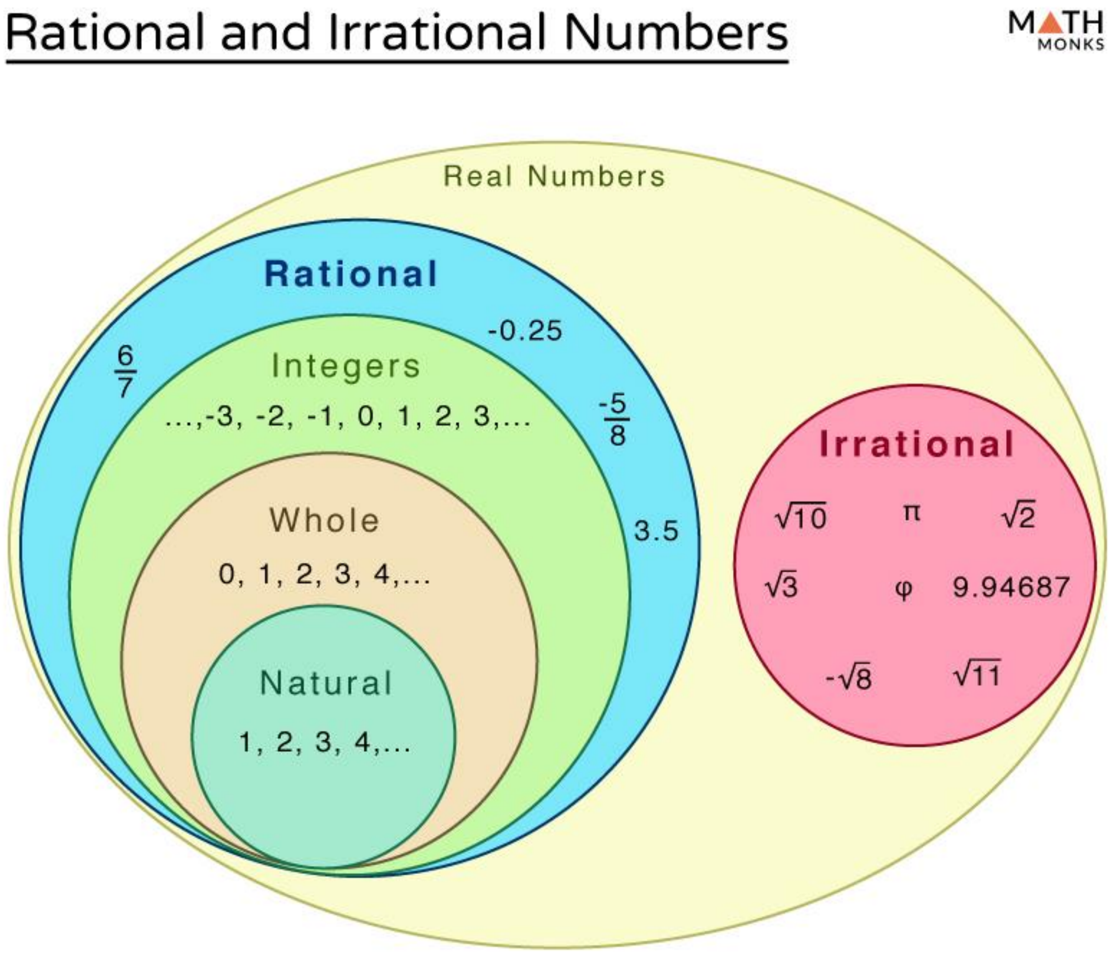
b. 0

c. -12.64039…

d. -½

e. 6.36

f. π

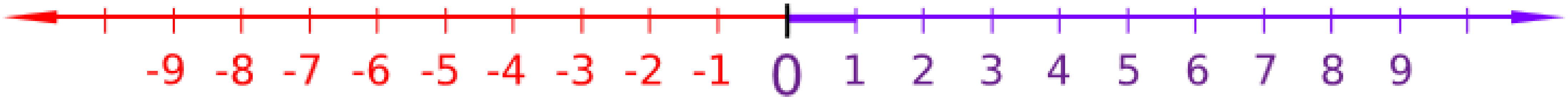


Real Number Line

A **number line** is a picture of a straight line on which every point is assumed to correspond to a real number and every real number to a point. A good way to get a picture of a set of numbers is to use a real number line.

To construct a number line,

1. Choose any point on a horizontal line and label it **0**.
2. Choose a point to the right of 0 and label it **1**.
3. The distance from 0 to 1 establishes a scale that can be used to locate more points, with positive numbers to the right of 0 and negative numbers to the left of 0.
4. The number 0 is neither positive nor negative.
5. Each number on a number line is called the ***coordinate*** of the point that it labels while the point is the ***graph*** of the number.



The properties of the real number system fall into three categories:

1) Algebraic properties

Say that the real numbers can be added, subtracted, multiplied and divided (except by 0)

A1) $a + (b + c) = (a + b) + c \; for \; all \; a, b, c$

A2) $a + b = b + a \; for \; all \; a, b$

A3) there is a number called "0" such that $a + 0 = a$

العنصر المحايد لعملية الجمع هو الصفر، اي ان اضافة اي عدد للصفر يعطي العدد نفسه.

M1) $a(b\,c) = (a\,b)c \;\; for \; all \; a, b, c$

M2) $ab = ba \;\; for \; all \; a, b$

M3) There is a number called "1" such that $a.1 = a$

العنصر المحايد لعملية الضرب هو 1، اي ان ضرب اي عدد في 1 يعطي نفس العدد.

D) $a(b + c) = ab + ac \;\; for \; all \; a, b, c$

A1 and M1 are associative laws. الخاصية التجميعية

A2 and M2 are commutative laws. الخاصية التبادلية

A3 and M3 are identity laws. خاصية العنصر المحايد

D is the distributive law. الخاصية التوزيعية

2) Order properties

The order properties of real numbers are

If a, b and c real numbers then:

a) $a < b \;\rightarrow a + c < b + c$ $\qquad (\rightarrow then)$

b) $a < b \;\rightarrow a - c < b - c$

c) $a < b \; and \; c > 0 \;\rightarrow ac < bc$

d) $a < b \; and \; c < 0 \rightarrow bc < ac$, $\boldsymbol{special\; case}: \boldsymbol{a < b \rightarrow -b < -a}$

e) $a > 0 \;\rightarrow \frac{1}{a} > 0$

f) If a and b are both positive or both negative, then $a < b \rightarrow \frac{1}{b} < \frac{1}{a}$

3) Completeness property. خاصية الإكتمال

Intuitively, completeness implies that there are not any "gaps" or "missing points" in the real number line.

Intervals

A subset of the real line is called an interval if it contains at least two numbers and contains all the real numbers lying between any two of its elements.

For example, the set of all real numbers x such that x>6 is an interval, as it is the set of all x such that $-2 \leq x \leq 5$.

Geometrically, intervals correspond to rays and line segments on the real line. There are two types of intervals, a finite and an infinite.

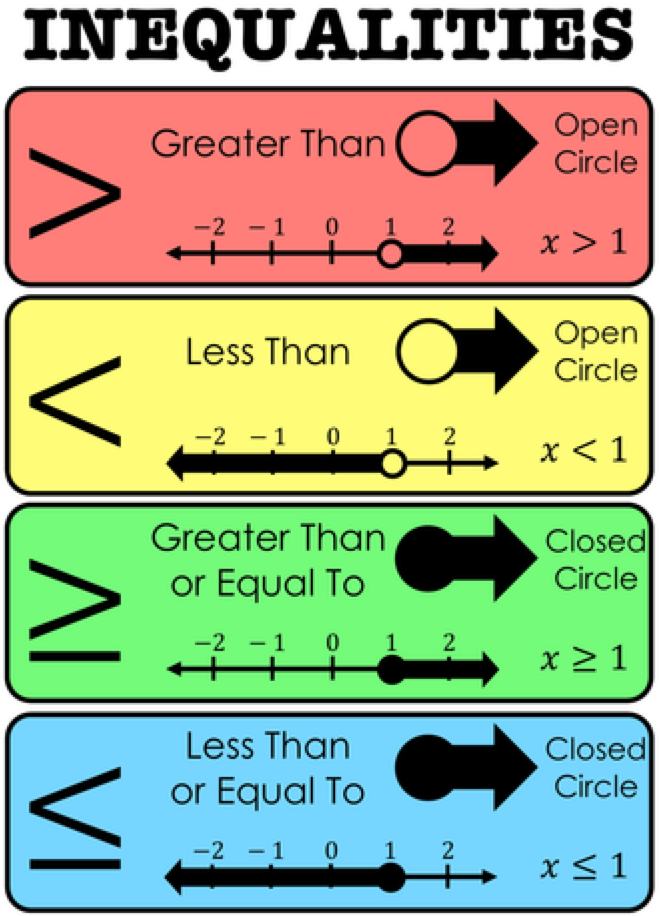
	Notation	Set description	Type	Picture
Finite	(a, b)	$\{x \mid a < x < b\}$	open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	closed	
	[a,b)	$\{x \mid a \leq x < b\}$	Half closed	
	(a,b]	$\{x \mid a < x \leq b\}$	Half closed	
Infinite	(a,∞)	$\{x \mid x > a\}$	open	
	[a,∞)	$\{x \mid x \geq a\}$	closed	
	(-∞,b)	$\{x \mid x < b\}$	open	
	(-∞,b]	$\{x \mid x \leq b\}$	closed	
	(-∞,∞)	ℝ(set of all real numbers)	Both open and closed	

Inequalities

When two numbers are not equal, one must be less than the other.

- the symbol < means "is less than", 8<9, -2<5, -9<-4
- the symbol > means "is greater than", 11>7, 2>-3
- the smaller of two numbers is always to the left of the other on a number line.

Inequalities on a Number line
a<b if a is to the left of b;
a>b if a is to the right of b;



Example:

Solve the following inequalities and show their solution sets on the real line.

$a)\ \ 2x - 1 < x + 3$

Solution:

$$2x - 1 < x + 3$$

$2x < x + 4$ $\quad$ *Add 1 to both sides*

$x < 4$ $\quad$ *Subtract x from both sides*

The solution set is the open interval $(-\infty, 4)$

a) $-\frac{x}{3} < 2x + 1$

Solution:

$$-\frac{x}{3} < 2x + 1$$

$-x < 6x + 3$ *Multiply both sides by 3*

$0 < 7x + 3$ *Add x to both sides*

$-3 < 7x$ *Subtract 3 from both sides*

$-\frac{3}{7} < x$ *Divide by 7*

The solution set is the open interval $\left(-\frac{3}{7}, \infty\right)$

b) $\frac{6}{x-1} \geq 5$

Solution:

$$\frac{6}{x-1} \geq 5$$

$6 \geq 5x - 5$ *Multiply both sides by* $(x-1)$

$11 \geq 5x$ *Add 5 to both sides*

$\frac{11}{5} \geq x$ *Divided by 5*

The inequality $\frac{6}{x-1} \geq 5$ *can hold only if*

$x > 1$, *because otherwise* $\frac{6}{x-1}$ *is undefined or negative.*

The solution set is the half − open interval $(1, \frac{11}{5}]$

c) $2x - 3 > 0$

Solution:

$2x - 3 > 0$ *Add 3 to both sides*

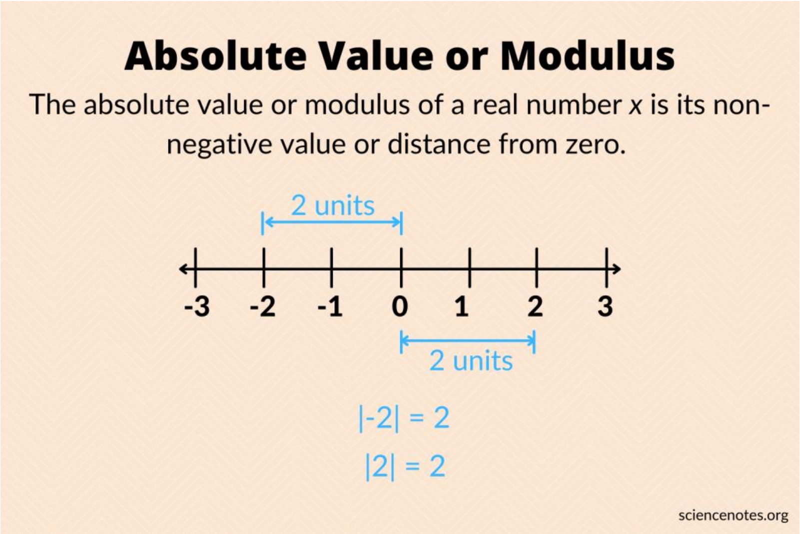
$2x > 3$ *Divided by 2*

$$x > \frac{3}{2}$$

The solution set is the open interval $\left(\frac{3}{2}, \infty\right)$

Absolute Value

The absolute value of a number *x*, denoted by **|*x*|**, is defined by the formula



$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Example:

|3|=3

|0|=0

|-5|=5

|-|a||=|a|

Geometrically, the absolute value of *x* is the distance from *x* to *0* on the real number line.

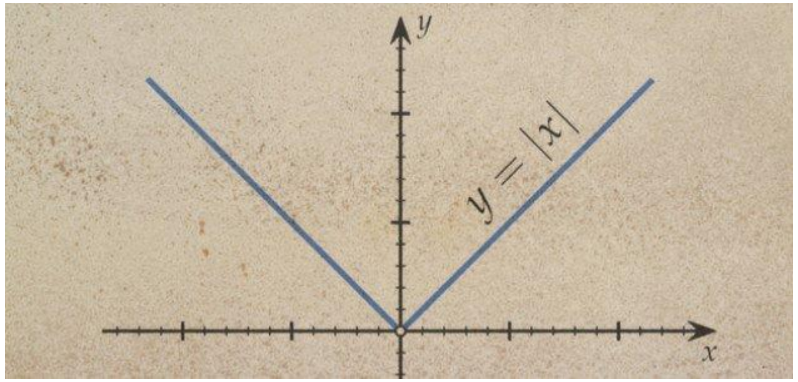
Absolute value properties

1) $|-a| = |a|$

2) $|ab| = |a||b|$

3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

4) $|a + b| \leq |a| + |b|$



Absolute Values and Intervals

The following statements are all consequences of the definition of absolute value and are often helpful when solving equation or inequalities involving absolute value.

If a is any positive number, then: $(\leftrightarrow \textit{if and only if})$

5) $|x| = a \leftrightarrow x = \pm a$

6) $|x| < a \leftrightarrow -a < x < a$

7) $|x| > a \leftrightarrow x > a \text{ or } x < -a$

8) $|x| \leq a \leftrightarrow -a \leq x \leq a$

9) $|x| \geq a \leftrightarrow x \geq a \text{ or } x \leq -a$

Example:

Solving equations with absolute value:

a) $|2x - 3| = 7$

Solution:

By property 5,

$$2x - 3 = \pm 7$$

So there are two possibilities:

$$2x - 3 = 7$$
$$2x = 10$$
$$x = 5$$

and

$$2x - 3 = -7$$
$$2x = -4$$
$$x = -2$$

The solution of $|2x - 3| = 7$ *are* $x = 5$ *and* $x = -2$

b) $\left|5-\frac{2}{x}\right|<1$

Solution:

By property 6

$$-1<5-\frac{2}{x}<1$$

$$-6<-\frac{2}{x}<-4 \quad subtract\ 5$$

$$3>\frac{1}{x}>2 \quad multiply\ by -\frac{1}{2}$$

$$\frac{1}{3}<x<\frac{1}{2} \quad take\ reciprocals$$

$$The\ solution\ set\ is\ the\ open\ interval\ \left(\frac{1}{3},\frac{1}{2}\right)$$

c) $|2x-3|\le 1$

Solution:

By property 8

$$-1\le 2x-3\le 1$$

$$2\le 2x\le 4 \quad add\ 3$$

$$1\le x\le 2 \quad divide\ by\ 2$$

$$The\ solution\ of\ |2x-3|\le 1\ is\ the\ closed\ interval\ [1,2]$$

d) $|2x-3|\ge 1$

Solution:

By property 9

$$2x-3\ge 1 \quad or \quad 2x-3\le -1$$

$$x-\frac{3}{2}\ge\frac{1}{2} \quad or \quad x-\frac{3}{2}\le-\frac{1}{2}$$

$$x\ge 2 \quad or \quad x\le 1$$

$$The\ solution\ set\ is\ (-\infty,1]\cup[2,\infty)$$

Quadratic Inequalities

A quadratic inequality (in one unknown) is an inequality that can be written in the form

$$ax^2 + bx + c < 0 \ (or > 0, or \ \leq 0, or \ \geq 0)$$

Where $a \neq 0$, $\quad n = 2$

1) $\boldsymbol{a > 0 \text{ and } b > 0 \ \rightarrow a * b > 0}$
2) $\boldsymbol{a < 0 \text{ and } b < 0 \ \rightarrow a * b > 0}$
3) $\boldsymbol{a > 0 \text{ and } b < 0 \ \rightarrow a * b < 0}$
4) $\boldsymbol{a * b > 0 \ \leftrightarrow (a > 0 \text{ and } b > 0) or \ (a < 0 \text{ and } b < 0)}$
5) $\boldsymbol{a * b < 0 \ \leftrightarrow (a > 0 \text{ and } b < 0) or \ (a < 0 \text{ and } b > 0)}$

Example

Find the solution set to inequalities

a. $\boldsymbol{12x^2 - x - 6 > 0}$

Solution:

$$12x^2 - x - 6 > 0$$
$$12x^2 - 9x + 8x - 6 = 0$$
$$3x(4x - 3) + 2(4x - 3) = 0$$
$$(4x - 3)(3x + 2) = 0$$
$$x = -\frac{2}{3}, \qquad x = \frac{3}{4}$$

$Note$: $\quad a * b > 0$
$\leftrightarrow (a > 0 \text{ and } b > 0) or \ (a < 0 \text{ and } b < 0)$

The solution set is $x < -\frac{2}{3} \quad and \quad x > \frac{3}{4}, \left(-\infty, -\frac{2}{3}\right) \cup \left(\frac{3}{4}, \infty\right)$

b- $\frac{2}{x+4} \geq \frac{1}{x+1}$

Solution:

$$\frac{2}{x+4} \geq \frac{1}{x+1}$$

$$\frac{2}{x+4} - \frac{1}{x+1} \geq 0$$

$$\frac{2(x+1)-(x+4)}{(x+4)(x+1)} \geq 0$$

$$\frac{2x+2-x-4}{(x+4)(x+1)} \geq 0$$

$$\frac{x-2}{(x+4)(x+1)} \geq 0$$

Examine the signs

x-2=0 if x=2 **x-2>0 if x>2** **x-2<0 if x<2**	**x+4=0 if x=-4** **x+4>0 if x>-4** **x+4<0 if x<-4**	**x+1=0 if x=-1** **x+1>0 if x>-1** **x+1<0 if x<-1**

Note that -4 and -1 are not in the solution set because they make the denominator zero.

The solution set is $(-4,-1) \cup [2,\infty)$

1) $x^2+6x+8<0$
2) $x^2+2x-35>0$
3) $x^2-9x+14\leq 0$
4) $x^2-x-30\geq 0$
5) $x^2>4(8-x)$
6) $3x^2-5x-1<4x^2+7x+19$
7) $2x^2+9x+10>0$
8) $7x^2-22x+16\leq 0$
9) $x^2-2x-24<0$
10) $12-5x\geq x+9$
11) $x^2-100>0$
12) $x^2+8x-105>0$
13) $x^2+2x-3\leq 0$
14) $x^2-x-12>0$
15) $-x^2+3x+10<0$
16) Given $A = x^2 - 7$ and $B = -4x + 5$, for what values of x is $A < B$?
17) When a projectile is fired into the air, its height h, in meters, t seconds later is given by the equation $h(t) = 11t - 3t^2$. When is the projectile at least 6 m above the ground?
18) The surface area, A, of a cylinder with radius r is given by the formula $A = 2r^2 - 5r$. What possible radii would result in an area that is greater than 12 cm^2 ?
19) When a baseball is hit by a batter, the height of the ball, $h(t)$, at time t, is determined by the equation $h(t) = -16t^2 + 64t + 4$. For which interval of time is the height of the ball greater than or equal to 52 feet?