Mathematics for medical physics

Functions

First lecture

Dr. Faten Monjed Hussein

الرياضيات للفيزياء الطبية

المحاضرة الأولى

المستوى الأول

Introduction:

1. The **natural numbers**, namely 1, 2, 3, 4,….
2. The whole numbers 0,1,2,…..
3. The **integers**, namely $0, \mp 1, \mp 2, \mp 3, \ldots$
4. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n , where *m* and *n* are integers and $n \neq 0$

Examples:1/2, -3/4, 57/1, 200/13…
The rational numbers are precisely the real numbers with decimal expansions that are either:

(a) Terminating (ending in an infinite string of zeros), for example,

(b) Eventually repeating (ending with a block of digits that repeats over and over), example:

23/11= 2.090909..= $2.\overline{09}$

3/4 = 0.75000، = 0.75

5. Irrational numbers. They are characterized by having nonterminating and nonrepeating decimal expansions

EX. $\sqrt{3}, \sqrt{5}, \pi$..

Real numbers that are rational **and** irrational numbers (all number groups)

The real numbers can be represented geometrically as points on a number line called the real line.

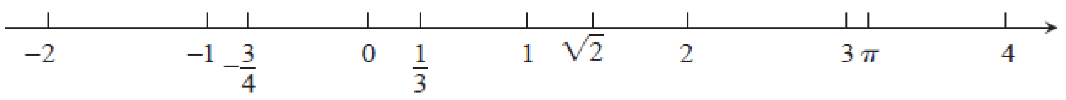


TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \ge 5$

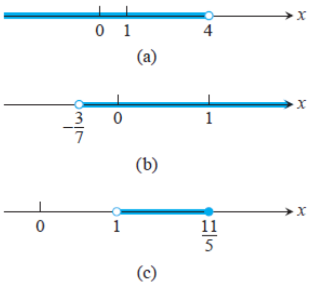


FIGURE 1.1 Solution sets for the inequalities in Example 1.

Solution

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \qquad \text{Add 1 to both sides.}$$

$$x < 4 \qquad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \qquad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \qquad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \qquad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \qquad \text{Divide by 7.}$$

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$

6. $|x| < a$ if and only if $-a < x < a$

7. $|x| > a$ if and only if $x > a$ or $x < -a$

8. $|x| \le a$ if and only if $-a \le x \le a$

9. $|x| \ge a$ if and only if $x \ge a$ or $x \le -a$

EXAMPLE 4 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{aligned} 2x - 3 &= 7 & \qquad 2x - 3 &= -7 \\ 2x &= 10 & 2x &= -4 \\ x &= 5 & x &= -2 \end{aligned}$$

Equivalent equations without absolute values

Solve as usual.

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 &\Leftrightarrow -1 < 5 - \frac{2}{x} < 1 && \text{Property 6} \\ &\Leftrightarrow -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow 3 > \frac{1}{x} > 2 && \text{Multiply by } -\tfrac{1}{2}. \\ &\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

solve the inequalities and show the solution sets on the real line.

5. $-2x > 4$

6. $8 - 3x \geq 5$

7. $5x - 3 \leq 7 - 3x$

8. $3(2 - x) > 2(3 + x)$

9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$

10. $\frac{6 - x}{4} < \frac{3x - 4}{2}$

11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$

12. $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$

14. $|y - 3| = 7$

15. $|2t + 5| = 4$

16. $|1 - t| = 1$

17. $|8 - 3s| = \frac{9}{2}$

18. $\left|\frac{s}{2} - 1\right| = 1$

Functions and Their Graphs

2. DEFINITION OF A FUNCTION

A function, defined for all numbers, is an association which to each number associates another number. If we denote a function by (f), then this association is denoted by:

$$x \to f(x)$$

We call ***f (x)*** the value of the function at x, or the **image** of ***x***. f(x) also can be written as:

$$y = f(x)$$

The set *D* of all possible input values is called the **domain** of the function, all values of $f(x)$ as x varies throughout D is called the **range of the function**.

A function can also be pictured as an **arrow diagram** (Figure 1). Each arrow associates an element of the domain D to a unique or single element in the set Y.

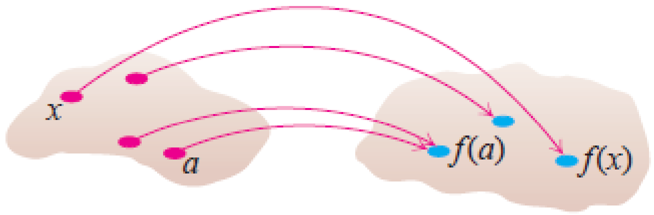


FIGURE 1.23 A function from a set D to a set Y assigns a unique element of Y to each element in D.

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Example 2

1. Linear function : $f: x \to ax + b$
2. Quadratic function: $f(x) \to ax^2 + bx + c$, where a,b,and c are constant and $a \neq 0$
3. Let G be the function: $\begin{cases} G(x) = 0 \text{ if } x \text{ is a rational number.} \\ G(x) = 1 \text{ if } x \text{ is not a rational number} \end{cases}$
4. $f: x \to \sqrt{f(x)}: f(x) \geq 0$

5. $f: x \to \frac{1}{f(x)} : f(x) \neq 0$

Graphs of Functions

A way to visualize a function is its graph. If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f.

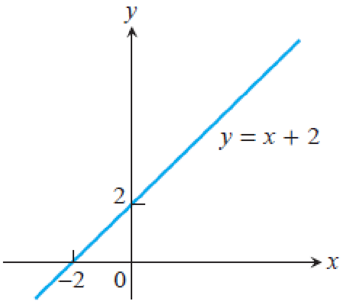


FIGURE 2 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

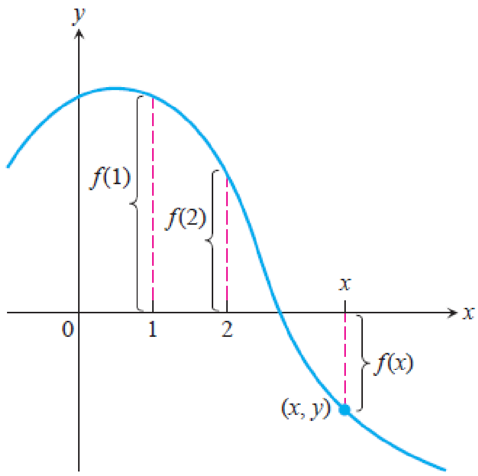
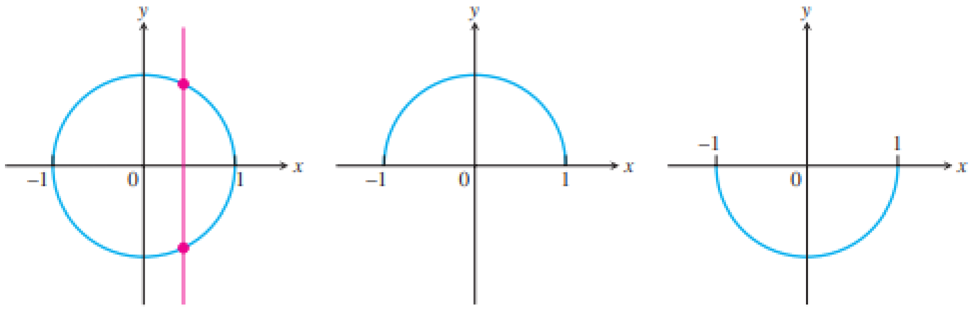


FIGURE 3 If (x, y) lies on the graph of f, then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

Not every curve in the coordinate plane can be the graph of a function. A function ƒ can have only one value for each x in its domain, so no vertical line can intersect the graph of a function more than once. If a is in the domain of the function ƒ, then the vertical line will intersect the graph of ƒ at the single point . A circle cannot be the graph of a function since some vertical lines intersect the circle twice. The circle in Figure 1.7a, however, does contain the graphs of two functions of x: the upper semicircle defined by the function and the lower semicircle defined by the function

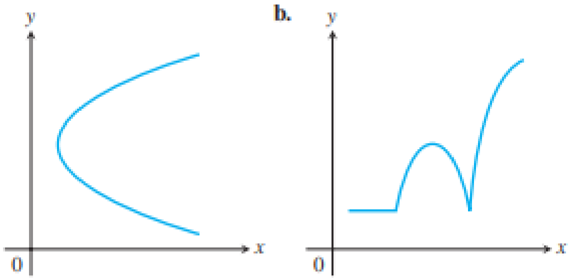


(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

7. a. **b.**



7. (a) Not the graph of a function of x since it fails the vertical line test.
   (b) Is the graph of a function of x since any vertical line intersects the graph at most once.

EXAMPLE 3

Graph the function $y = x^2$ over the interval [-2, +2]

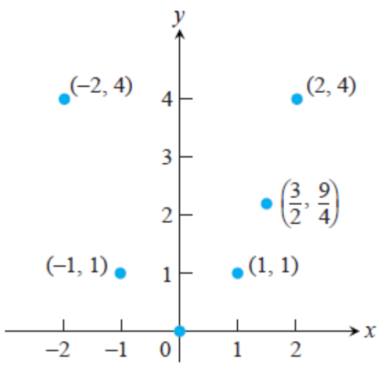
Solution

Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

2. Plot the points (x, y) whose coordinates appear in the table. Use fractions when they are convenient computationally.

3. Draw a smooth curve through the plotted points. Label the curve with its equation.



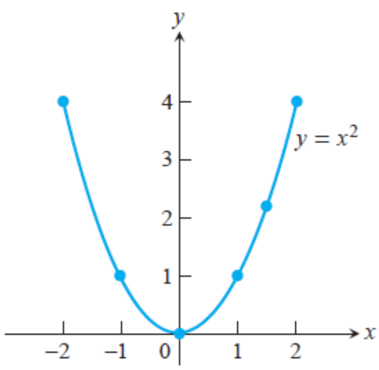


Fig. (5)

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Figure 1.29. Here are some other examples.

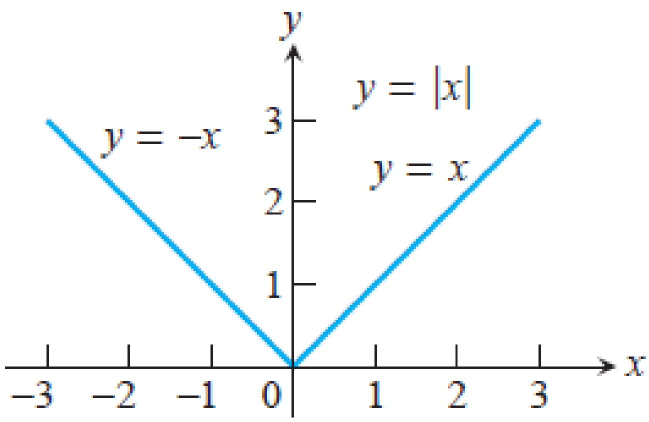


FIGURE (6) The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

EXAMPLE Graphing Piecewise-Defined Functions

The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x. The values of f are given by: $y = -x$ when $x < 0$, $y = x^2$ when $0 \le x \le 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.30). ■

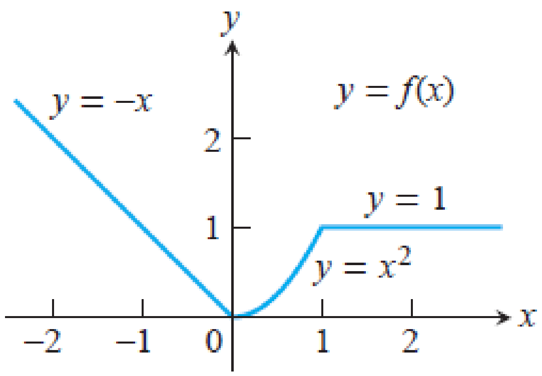


FIGURE 7 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 5).

EXAMPLE 1 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

EXAMPLE 2 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

$$\left|5 - \frac{2}{x}\right| < 1 \Leftrightarrow -1 < 5 - \frac{2}{x} < 1 \qquad \text{Property 6}$$

$$\Leftrightarrow -6 < -\frac{2}{x} < -4 \qquad \text{Subtract 5.}$$

$$\Leftrightarrow 3 > \frac{1}{x} > 2 \qquad \text{Multiply by } -\frac{1}{2}.$$

$$\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. \qquad \text{Take reciprocals.}$$

EXAMPLE Solve the inequality and show the solution set on the real line:

(a) $|2x - 3| \leq 1$ **(b)** $|2x - 3| \geq 1$

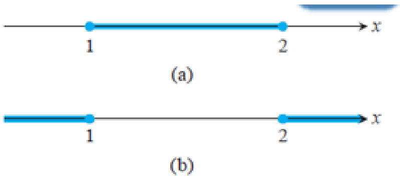


FIGURE 1.4 The solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 6.

Solution

(a)

$$|2x - 3| \leq 1$$

$$-1 \leq 2x - 3 \leq 1 \qquad \text{Property 8}$$

$$2 \leq 2x \leq 4 \qquad \text{Add 3.}$$

$$1 \leq x \leq 2 \qquad \text{Divide by 2.}$$

The solution set is the closed interval $[1, 2]$ (Figure 1.4a).

(b)

$$|2x - 3| \geq 1$$

$$2x - 3 \geq 1 \quad \text{or} \quad 2x - 3 \leq -1 \qquad \text{Property 9}$$

$$x - \frac{3}{2} \geq \frac{1}{2} \quad \text{or} \quad x - \frac{3}{2} \leq -\frac{1}{2} \qquad \text{Divide by 2.}$$

$$x \geq 2 \quad \text{or} \quad x \leq 1 \qquad \text{Add } \frac{3}{2}.$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b).

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

In Exercises 19–28, find the **(a)** domain and **(b)** range.

19. $y = |x| - 2$

20. $y = -2 + \sqrt{1 - x}$

21. $y = \sqrt{16 - x^2}$

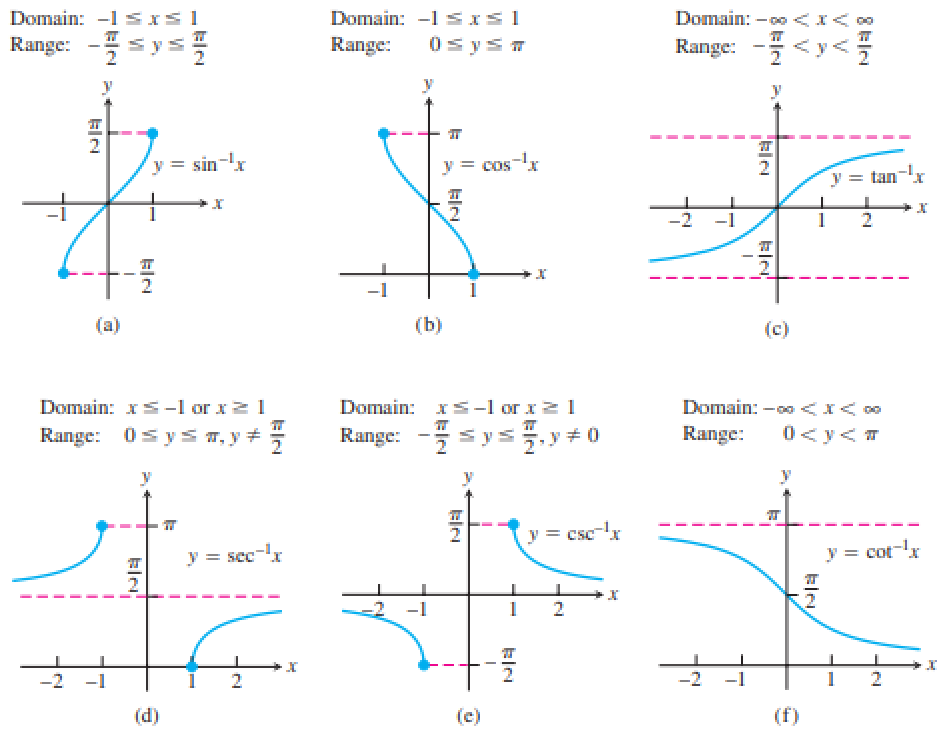
22. $y = 3^{2-x} + 1$

23. $y = 2e^{-x} - 3$

24. $y = \tan(2x - \pi)$

25. $y = 2\sin(3x + \pi) - 1$

26. $y = x^{2/5}$



Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

4. $g(x) = \sqrt{x^2 - 3x}$

5. $f(t) = \frac{4}{3 - t}$

6. $G(t) = \frac{2}{t^2 - 16}$

1. domain $= (-\infty, \infty)$; range $= [1, \infty)$

2. domain $= [0, \infty)$; range $= (-\infty, 1]$

3. domain $= [-2, \infty)$; y in range and $y = \sqrt{5x + 10} \geq 0 \Rightarrow$ y can be any positive real number $\Rightarrow$ range $= [0, \infty)$.

4. domain $= (-\infty, 0] \cup [3, \infty)$; y in range and $y = \sqrt{x^2 - 3x} \geq 0 \Rightarrow$ y can be any positive real number $\Rightarrow$ range $= [0, \infty)$.

5. domain $= (-\infty, 3) \cup (3, \infty)$; y in range and $y = \frac{4}{3-t}$, now if $t < 3 \Rightarrow 3 - t > 0 \Rightarrow \frac{4}{3-t} > 0$, or if $t > 3$ $\Rightarrow 3 - t < 0 \Rightarrow \frac{4}{3-t} < 0 \Rightarrow$ y can be any nonzero real number $\Rightarrow$ range $= (-\infty, 0) \cup (0, \infty)$.

6. domain $= (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$; y in range and $y = \frac{2}{t^2-16}$, now if $t < -4 \Rightarrow t^2 - 16 > 0 \Rightarrow \frac{2}{t^2-16} > 0$, or if $-4 < t < 4 \Rightarrow -16 \leq t^2 - 16 < 0 \Rightarrow -\frac{2}{16} \leq \frac{2}{t^2-16} < 0$, or if $t > 4 \Rightarrow t^2 - 16 > 0 \Rightarrow \frac{2}{t^2-16} > 0 \Rightarrow$ y can be any nonzero real number $\Rightarrow$ range $= (-\infty, -\frac{1}{8}] \cup (0, \infty)$.

Mathematics for medical physics

Types of function and Inverse function

Second lecture

Dr. Faten Monjed Hussein

الرياضيات للفيزياء الطبية
المستوى الأول
المحاضرة الثانية

Types of function and Inverse function

Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions.

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial.

$p(x) = ax^2 + bx + c$, are called **quadratic functions**.

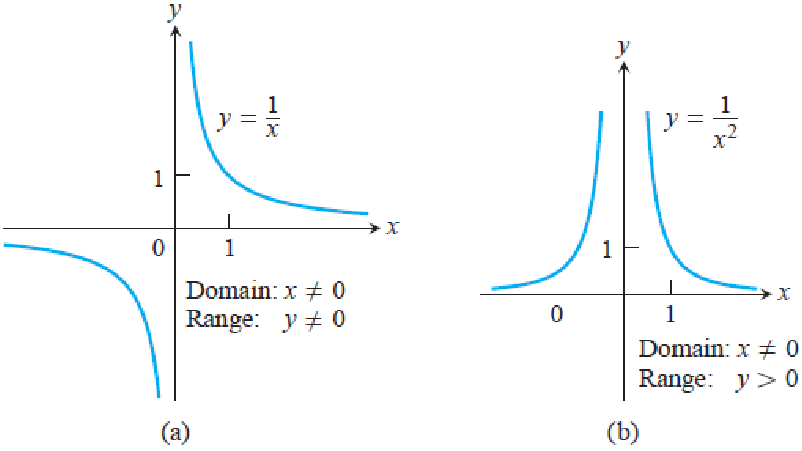
Rational Functions A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. For example, the function

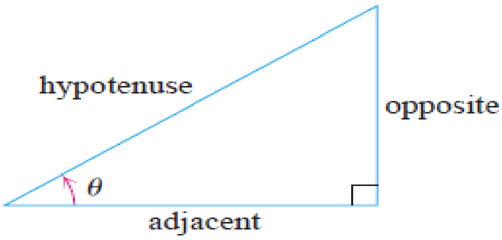
$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function with domain $\{x \mid x \neq -4/7\}$



Graphs of the power functions $f(x) = x^a$ for part (a) $a = -1$ and for part (b) $a = -2$.

Trigonometric function

 $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$	$\cos(A+B) = \cos A\cos B - \sin A\sin B$ $\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$ $1 + \cot^2\theta = \frac{1}{\sin^2\theta}$	$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

- Definitions

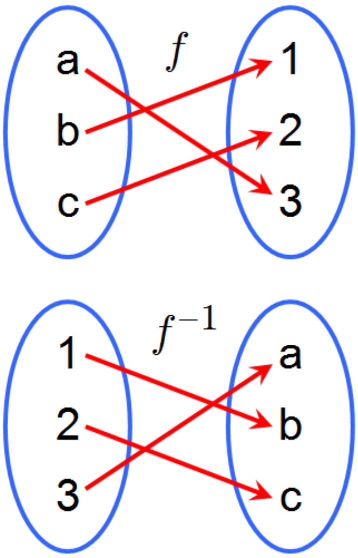
Let f be a function whose domain is the set X, and whose range is the set Y. Then f is *invertible* if there exists a function g from Y to X

- The **inverse function** of a function f (also called the **inverse** of f) is a function that undoes the operation of f. The inverse of f exists if and only if f is bijective, and if it exists, is denoted by f^{-1} .

For a function $f: x \to y \; then \; f^{-1}: y \to x$ its inverse

DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.
$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.



EXAMPLE Finding Trigonometric Function Values

If $\tan\theta = 3/2$ and $0 < \theta < \pi/2$, find the five other trigonometric functions of θ.

Solution From $\tan\theta = 3/2$, we construct the right triangle of height 3 (opposite) and base 2 (adjacent) in Figure 1.72. The Pythagorean theorem gives the length of the hypotenuse, $\sqrt{4+9} = \sqrt{13}$. From the triangle we write the values of the other five trigonometric functions:

$$\cos\theta = \frac{2}{\sqrt{13}}, \quad \sin\theta = \frac{3}{\sqrt{13}}, \quad \sec\theta = \frac{\sqrt{13}}{2}, \quad \csc\theta = \frac{\sqrt{13}}{3}, \quad \cot\theta = \frac{2}{3}$$

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$$\cos(\theta + 2\pi) = \cos\theta \qquad \sin(\theta + 2\pi) = \sin\theta \qquad \tan(\theta + 2\pi) = \tan\theta$$
$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta \qquad \cot(\theta + 2\pi) = \cot\theta$$

Similarly, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

Ex:

Prof that $cos(\theta + 2\pi) = cos(\theta)$

$$cos(\theta + 2\pi) = cos(\theta)cos(2\pi) - sin(\theta)sin(2\pi)$$

$$cos(\theta + 2\pi) = cos(\theta) - 0 = cos(\theta)$$

EXAMPLE 8 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.69a.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.69b. ■

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1} x$	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

EX:

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \in \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

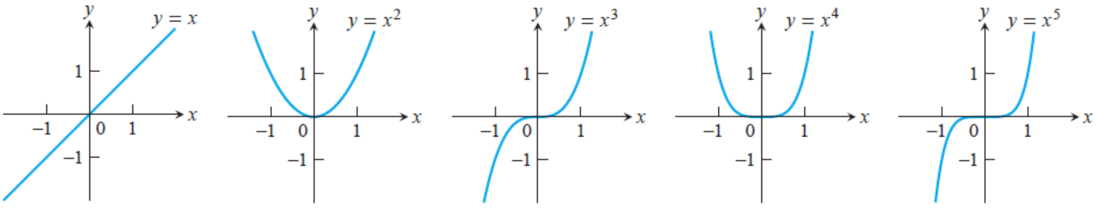
11. $\tan x = \frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

12. $\sin x = -\frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

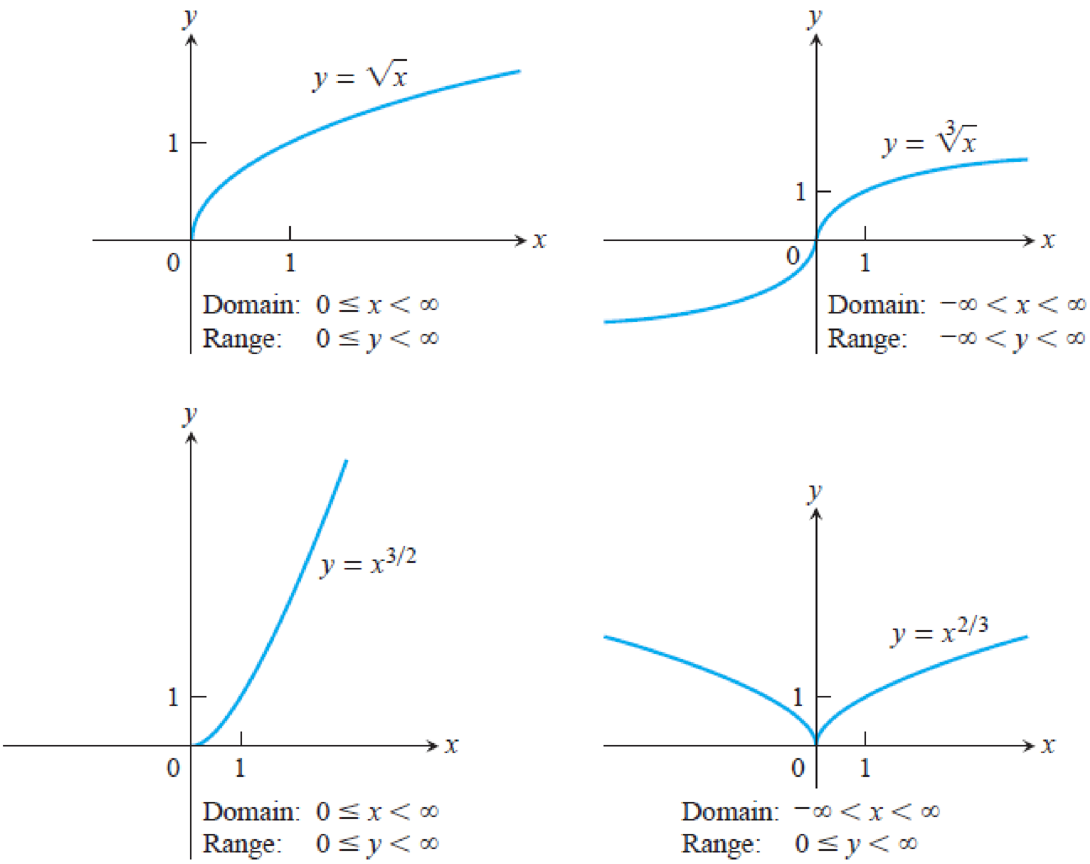
Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.36. These functions are defined for all real values of x. Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval $(-1, 1)$, and also rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin.



Graphs of $f(x) = x^n, n = 1, 2, 3, 4, 5$ defined for $-\infty < x < \infty$.



Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

DEFINITION The **logarithm function with base a**, $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

The domain of $\log_a x$ is $(0, \infty)$, the range of a^x. The range of $\log_a x$ is $(-\infty, \infty)$, the domain of a^x.

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**. For the natural logarithm,

$$\ln x = y \quad \Leftrightarrow \quad e^y = x.$$

In particular, if we set $x = e$, we obtain

$$\ln e = 1$$

because $e^1 = e$.

Properties of Logarithms

THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

Inverse Properties for a^x and $\log_a x$
1. Base a: $a^{\log_a x} = x$, $\log_a a^x = x$, $a > 0, a \neq 1, x > 0$
2. Base e: $e^{\ln x} = x$, $\ln e^x = x$, $x > 0$

EXAMPLE 1 Recognizing Functions

Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category. For example, $f(x) = x^2$ is both a power function and a polynomial of second degree.

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ **(b)** $g(x) = 7^x$ **(c)** $h(z) = z^7$

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$

Solution

(a) $f(x) = 1 + x - \frac{1}{2}x^5$ is a polynomial of degree 5.

(b) $g(x) = 7^x$ is an exponential function with base 7. Notice that the variable x is the exponent.

(c) $h(z) = z^7$ is a power function. (The variable z is the base.)

(d) $y(t) = \sin\left(t - \frac{\pi}{4}\right)$ is a trigonometric function. ■

Example:, consider the real-valued function of a real variable given by:

$$f(x) = 5x - 7.$$

Inverse of (f) is the function: $X = \left(\frac{f(x)+7}{5}\right)$

Mathematics for medical physics

LIMITS AND CONTINUITY

Third lecture

Dr. Faten Monjed Hussein

الرياضيات للفيزياء الطبية
المستوى الأول
المحاضرة الثالثة

LIMITS AND CONTINUITY

Average Rates of Change and Secant Lines

Given an arbitrary function y=f(x), we calculate the average rate of change of y with respect to x over the interval[x_1, x_2] by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$,

by the length of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

$$h \neq 0$$

Geometrically, the rate of change of *f* over [x_1, x_2], is the slope of the line through the points p(x_1,f(x_1)), and Q(x_2, f(x_2a)) (Figure 2.1).

In geometry, a line joining two points of a curve is a secant to the curve. Thus, the average rate of change of *f* from x_1 to x_2 is identical with the slope of secant PQ.

Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero.

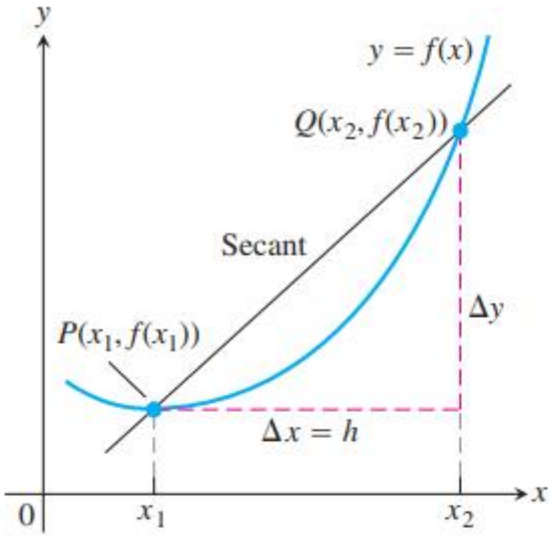


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

what is meant by the slope of a curve at a point P on the curve?

If there is a tangent line to the curve at P—a line that just touches the curve like the tangent to a circle—it **would be reasonable to identify the slope of the tangent as the slope of the curve at P**.

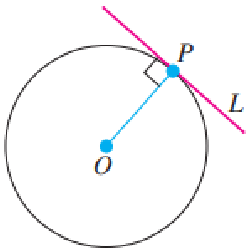


FIGURE 2.2 *L* is tangent to the circle at *P* if it passes through *P* perpendicular to radius *OP*.

EXAMPLE 3: Find the slope of the parabola $y = x^2$ at the point P(2, 4). Write an equation for the tangent to the parabola at this point.

Solution:

We begin with a secant line through P(2, 4) and Q(2+h, $(2+h)^2$). We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$secant\ slope = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - (2^2)}{h} = \frac{h^2 + 4h + 4 - 4}{h} = h + 4$$

If $h > 0$ then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$ then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero, and the secant slope approaches 4. We take 4 to be the parabola's slope at P.

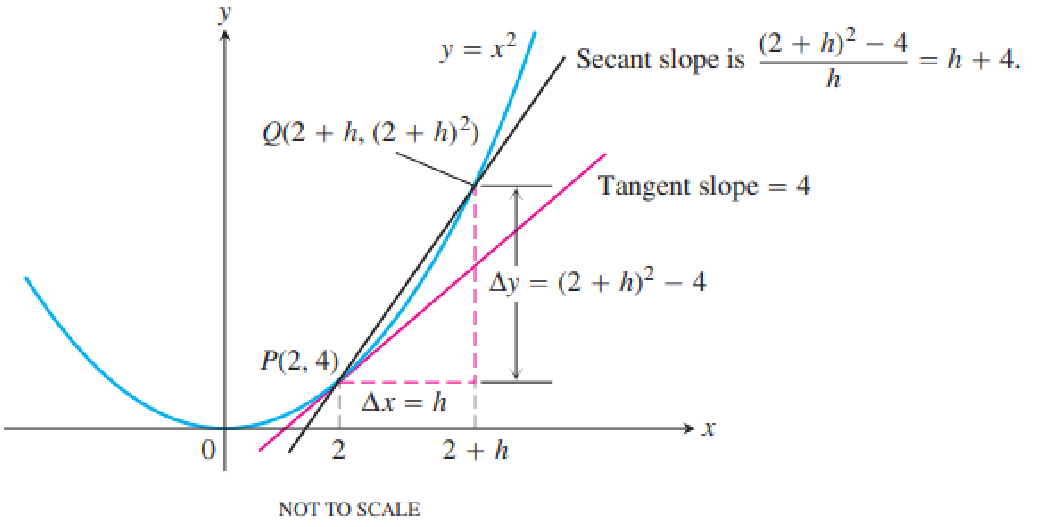


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2)$$

$$y = 4x - 4$$

The instantaneous rates were found to be the values of the average rates of change, as the interval of length h approached 0. That is, the instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero.

The average rate of change corresponds to the slope of a secant line;

the instantaneous rate corresponds to the slope of the tangent line as the independent variable approaches a fixed value.

Examples: Find the average rate of change of the function over the given interval or intervals:

1) $f(x) = x^3 + 1$ a) [2,3] b)[-1,+1]
2) $g(x) = x^2$ a)[-1,+1] b)[-2,0]
3) $h(t) = \cot(t)$ a)$[\frac{\pi}{4}, \frac{3\pi}{4}]$ b) $[\frac{\pi}{6}, \frac{\pi}{2}]$

4) $k(t) = \cot(t) + 2$ a)$[0, \pi]$, b)$[-\pi, \pi]$

Solution:

$$1).\ a)\ \frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2} = \frac{28 - 9}{1} = 19,$$

$$b)\ \frac{\Delta f}{\Delta x} = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$$

$$2).\ a)\ \frac{\Delta g}{\Delta x} = \frac{g(1) - g(-1)}{1 - (-1)} = \frac{281 - 1 - 9}{2} = 0,$$

$$b)\ \frac{\Delta g}{\Delta x} = \frac{g(0) - g(-2)}{0 - (-2)} = \frac{0 - 4}{2} = -2$$

$$3).\ a)\ \frac{\Delta h}{\Delta x} = \frac{h\left(3\frac{\pi}{4}\right) - h\left(\frac{\pi}{4}\right)}{3\frac{\pi}{4} - \left(\frac{\pi}{4}\right)} = \frac{-1 - 1}{\frac{\pi}{2}} = -\frac{4}{\pi},$$

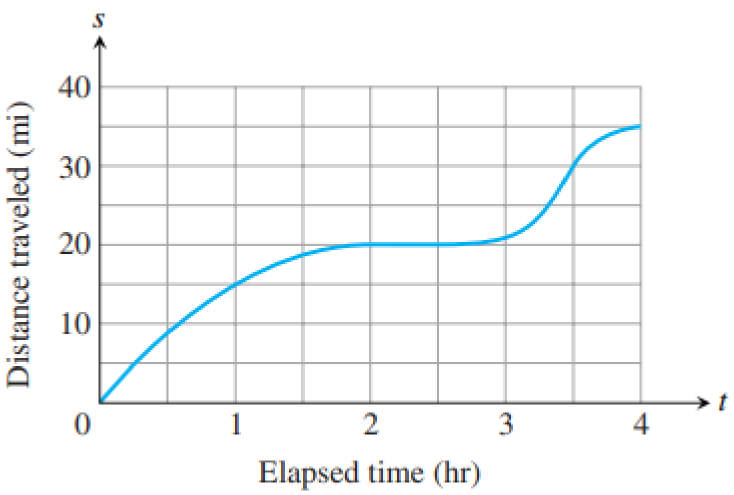
$$b)\ \frac{\Delta h}{\Delta x} = \frac{h\left(\frac{\pi}{2}\right) - h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \left(\frac{\pi}{6}\right)} = \frac{0 - \sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$$

$$4).\ a)\ \frac{\Delta k}{\Delta x} = \frac{k(\pi) - k(0)}{\pi - 0} = \frac{1 - 3}{\pi} = -\frac{2}{\pi},$$

$$b)\ \frac{\Delta k}{\Delta x} = \frac{k(\pi) - k(-\pi)}{\pi - (-\pi)} = \frac{3 - 1}{2\pi} = \frac{1}{\pi}$$

Examples

The accompanying graph shows the total distance s traveled by a bicyclist after (t) hours.



a. Estimate the bicyclist's average speed over the time intervals [0, 1], [1, 2.5], and [2.5, 3.5].
b. Estimate the bicyclist's instantaneous speed at the times t=1/2, t=2 and t=3 .
c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

Solution:

a) [0,1] $\frac{\Delta x}{\Delta t} = \frac{15-0}{1-0} = 15\ mi/h$, [1,1.25] $\frac{\Delta x}{\Delta t} = \frac{20-15}{25-1} = \frac{10}{3} mi/h,$

[2.5,3.5] $\frac{\Delta x}{\Delta t} = \frac{30-20}{3.5-2.5} = 10 mi/\text{h}$

b) At p(1/2,7.5) since the portion of the graph from t=0 tot=1 is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at t=1/2 is $v = \frac{15-7.5}{1-0.5} = 15\ mi/h$
at p(2,20) since the portion of the graph from t=2s to t=2.5s is nearly linear, instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at t=1/2 is $v = \frac{20-20}{2.5-2} = 0\ mi/h.$
for values of t less than 2, we have:

Q	Slope of PQ = $\frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at t = 2 is 0 mi/hr.
At P(3, 22):

Q	Slope of PQ = $\frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr

Q	Slope of PQ = $\frac{\Delta s}{\Delta t}$
$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at t = 3 is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at t = 3.5. Thus for P(3.5, 30)

Q	Slope of PQ = $\frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr

Q	Slope of PQ = $\frac{\Delta s}{\Delta t}$
$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at t = 3.5 is about 20 mi/hr.

Limit of a Function and Limit Laws:

limits of Function Values:

when seeking the instantaneous rate of change in y by considering the quotient function for h closer and closer to zero.

EXAMPLE :

How does the function:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

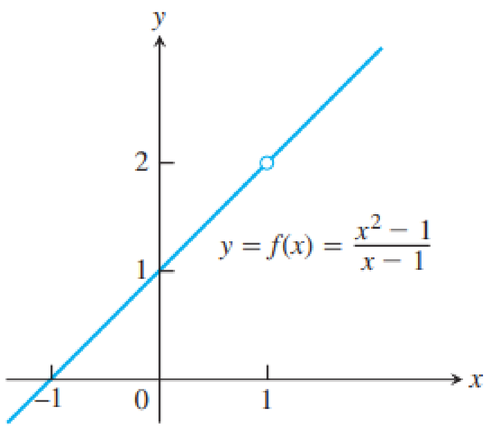
behave near x=1?

Solution:

The given formula defines f for all real numbers x except x=1 (we cannot divide by zero). For any$x \neq 1$. we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{(x-1)} = x+1$$

The graph of f is $y = x+1$ the line with the point (1, 2) removed. This removed point is shown as a "hole" in Figure.



Even though f(1) is not defined, it is clear that we can make the value of f(x) as close as we want to 2 by choosing x close enough to 1 (Table 2.2).

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Let's generalize the idea illustrated in Example 1.

Suppose $f(x)$ is defined on an open interval about x_0 , except x_0 possibly at itself. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to we say that f approaches the **limit L** as **x** approaches x_0 and write:

$$\lim_{x \to x_0} f(x) = L$$

"the limit of f(x) as x approaches x_0 is L."

$$\lim_{x \to 1} f(x) = 2$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

EXAMPLE This example illustrates that the limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$.

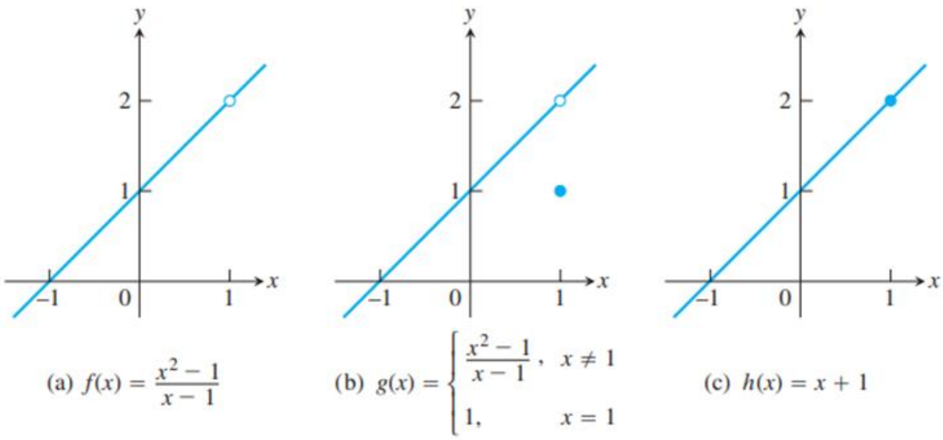
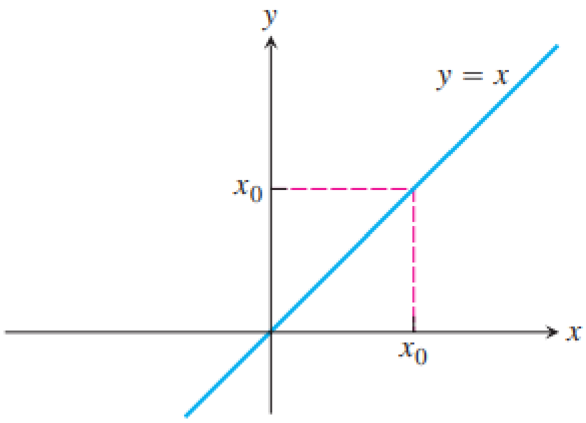
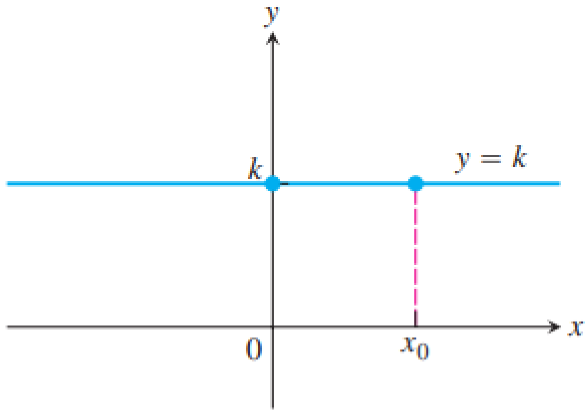


FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value is significant, and we return to it in Section 2.5. ■



(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points x_0.

EXAMPLE

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.9a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.9b),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

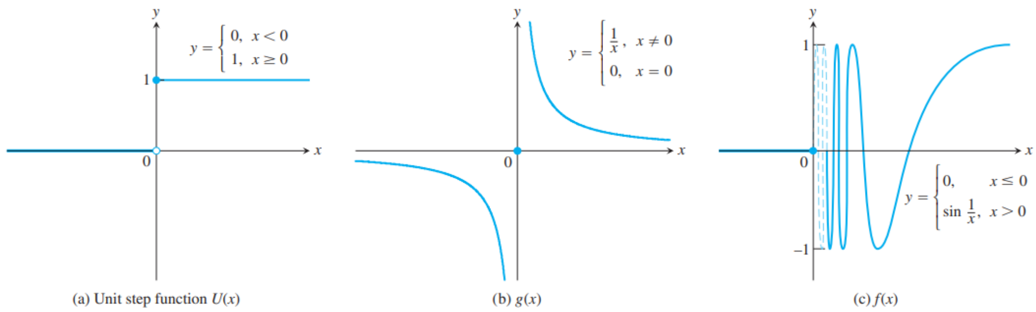


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

The Limit Laws:

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

EXAMPLE Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 3) a
the properties of limits to find the following limits.

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a)

$$\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 \qquad \text{Sum and Difference Rules}$$

$$= c^3 + 4c^2 - 3 \qquad \text{Power and Multiple Rules}$$

(b)

$$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} \qquad \text{Quotient Rule}$$

$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} \qquad \text{Sum and Difference Rules}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \qquad \text{Power or Product Rule}$$

(c)

$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)} \qquad \text{Root Rule with } n = 2$$

$$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} \qquad \text{Difference Rule}$$

$$= \sqrt{4(-2)^2 - 3} \qquad \text{Product and Multiple Rules}$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

The Sandwich Theorem

The sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem:

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.13).

EXAMPLE The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin \theta = 0$ **(b)** $\lim_{\theta \to 0} \cos \theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \leq \sin \theta \leq |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta \to 0} (-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin \theta = 0.$$

(b) From Section 1.3, $0 \leq 1 - \cos \theta \leq |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta \to 0} (1 - \cos \theta) = 0$ or

$$\lim_{\theta \to 0} \cos \theta = 1.$$

(c) Since $-|f(x)| \leq f(x) \leq |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$. ■

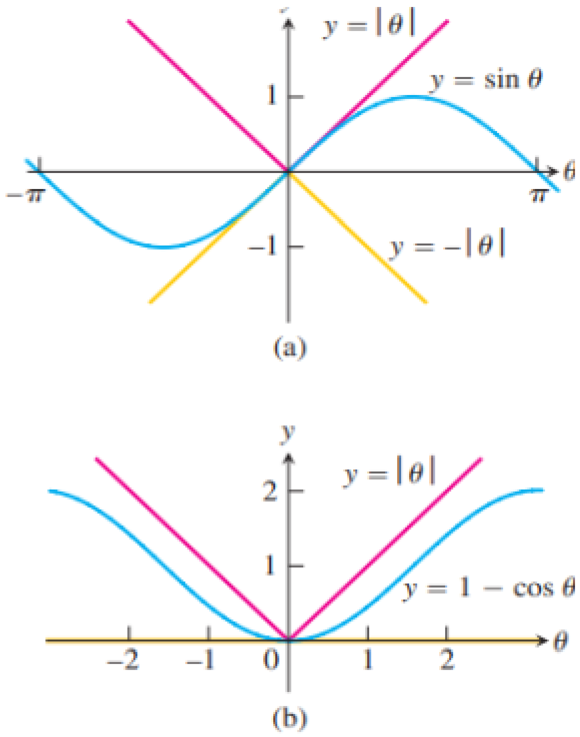


FIGURE The Sandwich Theorem confirms the limits in Example 11.

Another important property of limits is given by the next theorem. A proof is given in the next section.

> **THEOREM 5** If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then
>
> $$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Example:

Find the limits in Exercises 11–22.

11. $\lim_{x\to -7}(2x+5)$

12. $\lim_{x\to 2}(-x^2+5x-2)$

13. $\lim_{t\to 6} 8(t-5)(t-7)$

14. $\lim_{x\to -2}(x^3-2x^2+4x+8)$

15. $\lim_{x\to 2}\dfrac{x+3}{x+6}$

16. $\lim_{s\to 2/3} 3s(2s-1)$

17. $\lim_{x\to -1} 3(2x-1)^2$

18. $\lim_{y\to 2}\dfrac{y+2}{y^2+5y+6}$

19. $\lim_{y\to -3}(5-y)^{4/3}$

20. $\lim_{z\to 0}(2z-8)^{1/3}$

21. $\lim_{h\to 0}\dfrac{3}{\sqrt{3h+1}+1}$

22. $\lim_{h\to 0}\dfrac{\sqrt{5h+4}-2}{h}$

11. $\lim_{x\to -7}(2x+5)=2(-7)+5=-14+5=-9$

12. $\lim_{x\to 2}(-x^2+5x-2)=-(2)^2+5(2)-2=-4+10-2=4$

13. $\lim_{t\to 6} 8(t-5)(t-7)=8(6-5)(6-7)=-8$

14. $\lim_{x\to -2}(x^3-2x^2+4x+8)=(-2)^3-2(-2)^2+4(-2)+8=-8-8-8+8=-16$

15. $\lim_{x\to 2}\frac{x+3}{x+6}=\frac{2+3}{2+6}=\frac{5}{8}$

16. $\lim_{s\to \frac{2}{3}} 3s(2s-1)=3\left(\frac{2}{3}\right)\left[2\left(\frac{2}{3}\right)-1\right]=2\left(\frac{4}{3}-1\right)=\frac{2}{3}$

17. $\lim_{x\to -1} 3(2x-1)^2=3(2(-1)-1)^2=3(-3)^2=27$

18. $\lim_{y\to 2}\frac{y+2}{y^2+5y+6}=\frac{2+2}{(2)^2+5(2)+6}=\frac{4}{4+10+6}=\frac{4}{20}=\frac{1}{5}$

19. $\lim_{y\to -3}(5-y)^{4/3}=[5-(-3)]^{4/3}=(8)^{4/3}=\left((8)^{1/3}\right)^4=2^4=16$

20. $\lim_{z\to 0}(2z-8)^{1/3}=(2(0)-8)^{1/3}=(-8)^{1/3}=-2$

21. $\lim_{h\to 0}\frac{3}{\sqrt{3h+1}+1}=\frac{3}{\sqrt{3(0)+1}+1}=\frac{3}{\sqrt{1}+1}=\frac{3}{2}$

22. $\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}=\lim_{h\to 0}\frac{\sqrt{5h+4}-2}{h}\cdot\frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2}=\lim_{h\to 0}\frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5h}{h\left(\sqrt{5h+4}+2\right)}=\lim_{h\to 0}\frac{5}{\sqrt{5h+4}+2}$
$=\frac{5}{\sqrt{4}+2}=\frac{5}{4}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x \to 5} \frac{x-5}{x^2-25}$

24. $\lim_{x \to -3} \frac{x+3}{x^2+4x+3}$

25. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5}$

26. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2}$

27. $\lim_{t \to 1} \frac{t^2+t-2}{t^2-1}$

28. $\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2}$

29. $\lim_{x \to -2} \frac{-2x-4}{x^3+2x^2}$

30. $\lim_{y \to 0} \frac{5y^3+8y^2}{3y^4-16y^2}$

31. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1}$

32. $\lim_{x \to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

33. $\lim_{u \to 1} \frac{u^4-1}{u^3-1}$

34. $\lim_{v \to 2} \frac{v^3-8}{v^4-16}$

35. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

36. $\lim_{x \to 4} \frac{4x-x^2}{2-\sqrt{x}}$

37. $\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$

38. $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

39. $\lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

40. $\lim_{x \to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

41. $\lim_{x \to -3} \frac{2-\sqrt{x^2-5}}{x+3}$

42. $\lim_{x \to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

Solution:

23. $\lim_{x \to 5} \frac{x-5}{x^2-25} = \lim_{x \to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x \to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$

24. $\lim_{x \to -3} \frac{x+3}{x^2+4x+3} = \lim_{x \to -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x \to -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$

25. $\lim_{x \to -5} \frac{x^2+3x-10}{x+5} = \lim_{x \to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x \to -5} (x-2) = -5-2 = -7$

26. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2} = \lim_{x \to 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x \to 2} (x-5) = 2-5 = -3$

27. $\lim_{t \to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t \to 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \to 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

28. $\lim_{t \to -1} \frac{t^2+3t+2}{t^2-t-2} = \lim_{t \to -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t \to -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

29. $\lim_{x \to -2} \frac{-2x-4}{x^3+2x^2} = \lim_{x \to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

30. $\lim_{y \to 0} \frac{5y^3+8y^2}{3y^4-16y^2} = \lim_{y \to 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \to 0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$

31. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x \to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x \to 1} \left(\frac{1-x}{x} \cdot \frac{1}{x-1}\right) = \lim_{x \to 1} -\frac{1}{x} = -1$

32. $\lim_{x \to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x \to 1} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim_{x \to 1} \left(\frac{2x}{(x-1)(x+1)} \cdot \frac{1}{x}\right) = \lim_{x \to 1} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$

33. $\lim_{u \to 1} \frac{u^4-1}{u^3-1} = \lim_{u \to 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u \to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

34. $\lim_{v \to 2} \frac{v^3-8}{v^4-16} = \lim_{v \to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v \to 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$

35. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

36. $\lim_{x \to 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x \to 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x \to 4} x(2+\sqrt{x}) = 4(2+2) = 16$

37. $\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x \to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)} = \lim_{x \to 1} \frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} = \lim_{x \to 1} \left(\sqrt{x+3}+2\right) = \sqrt{4}+2 = 4$

38. $\lim_{x \to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x \to -1} \frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x \to -1} \frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim_{x \to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

39. $\lim_{x \to 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x \to 2} \frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)}$

$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x \to 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

40. $\lim_{x \to -2} \frac{x+2}{\sqrt{x^2+5}-3} = \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{\left(\sqrt{x^2+5}-3\right)\left(\sqrt{x^2+5}+3\right)} = \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x^2+5)-9}$

$= \lim_{x \to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x+2)(x-2)} = \lim_{x \to -2} \frac{\sqrt{x^2+5}+3}{x-2} = \frac{\sqrt{9}+3}{-4} = -\frac{3}{2}$

51. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0}2f(x)-\lim_{x\to 0}g(x)}{\left(\lim_{x\to 0}\left(f(x)+7\right)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2\lim_{x\to 0}f(x)-\lim_{x\to 0}g(x)}{\left(\lim_{x\to 0}f(x)+\lim_{x\to 0}7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

52. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \qquad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1}5h(x)}}{\left(\lim_{x\to 1}p(x)\right)\left(\lim_{x\to 1}\left(4-r(x)\right)\right)} \qquad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1}h(x)}}{\left(\lim_{x\to 1}p(x)\right)\left(\lim_{x\to 1}4-\lim_{x\to 1}r(x)\right)} \qquad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

53. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim_{x\to c} f(x)g(x)$ **b.** $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c} (f(x)+3g(x))$ **d.** $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

54. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

a. $\lim_{x\to 4} (g(x)+3)$ **b.** $\lim_{x\to 4} xf(x)$

c. $\lim_{x\to 4} (g(x))^2$ **d.** $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

55. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim_{x\to b} (f(x)+g(x))$ **b.** $\lim_{x\to b} f(x)\cdot g(x)$

c. $\lim_{x\to b} 4g(x)$ **d.** $\lim_{x\to b} f(x)/g(x)$

56. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

a. $\lim_{x\to -2} (p(x)+r(x)+s(x))$

b. $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

The Derivative of a Function

Dr. Faten Monjed Hussein

Mathematics for medical physics

الرياضيات للفيزياء الطبية

الملزمة الرابعة

المستوى الأول

Dr. F. M. Alfeel

3-1 Tangents and the Derivative at a Point:

3-1-1Finding a Tangent to the Graph of a Function:

To find a tangent to an arbitrary curve y=f(x), at a point p(x_0, f(x_0)), we calculate the slope of the secant through P and a nearby point Q(x_0+h,f(x_0 +h)). We then investigate the limit of the slope as h $\longrightarrow$0 (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

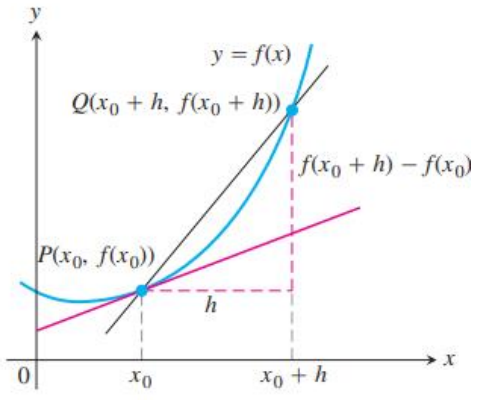


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

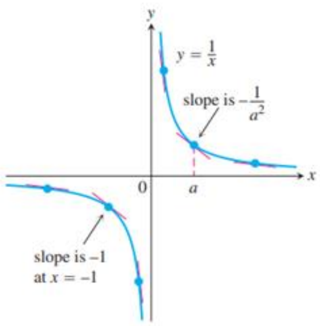


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0}\frac{-h}{ha(a+h)} = \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$\lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

Our original estimate of 32 ft/sec in Section 2.1 was right. ■

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point. All of these ideas refer to the same limit.

> The following are all interpretations for the limit of the difference quotient,
>
> $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
>
> 1. The slope of the graph of $y = f(x)$ at $x = x_0$
> 2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
> 3. The rate of change of $f(x)$ with respect to x at $x = x_0$
> 4. The derivative $f'(x_0)$ at a point

In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2, \quad (-1, 3)$

6. $y = (x - 1)^2 + 1, \quad (1, 1)$

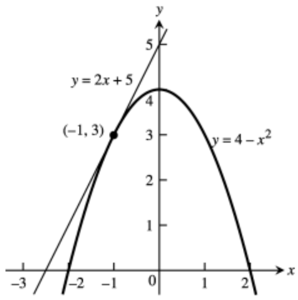
7. $y = 2\sqrt{x}, \quad (1, 2)$

8. $y = \dfrac{1}{x^2}, \quad (-1, 1)$

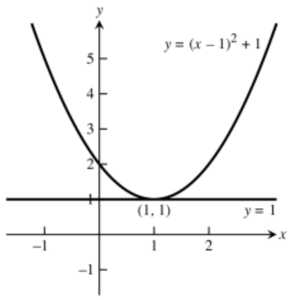
9. $y = x^3, \quad (-2, -8)$

10. $y = \dfrac{1}{x^3}, \quad \left(-2, -\dfrac{1}{8}\right)$

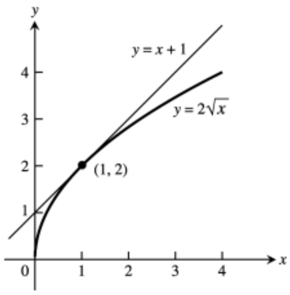
5. $m = \lim\limits_{h \to 0} \frac{[4-(-1+h)^2]-(4-(-1)^2)}{h}$
$= \lim\limits_{h \to 0} \frac{-(1-2h+h^2)+1}{h} = \lim\limits_{h \to 0} \frac{h(2-h)}{h} = 2;$
at $(-1, 3)$: $y = 3 + 2(x-(-1)) \Rightarrow y = 2x + 5$, tangent line



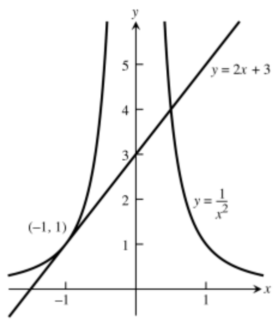
6. $m = \lim\limits_{h \to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h} = \lim\limits_{h \to 0} \frac{h^2}{h}$
$= \lim\limits_{h \to 0} h = 0$; at $(1, 1)$: $y = 1 + 0(x-1) \Rightarrow y = 1$, tangent line



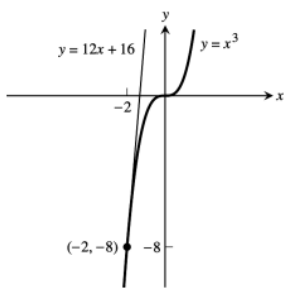
7. $m = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$
$= \lim\limits_{h \to 0} \frac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{1+h}+1} = 1;$
at $(1, 2)$: $y = 2 + 1(x-1) \Rightarrow y = x + 1$, tangent line



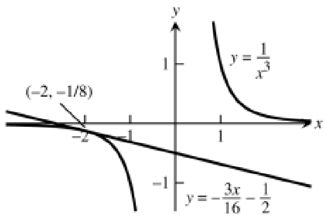
8. $m = \lim\limits_{h \to 0} \frac{\frac{1}{(-1+h)^2}-\frac{1}{(-1)^2}}{h} = \lim\limits_{h \to 0} \frac{1-(-1+h)^2}{h(-1+h)^2}$
$= \lim\limits_{h \to 0} \frac{-(-2h+h^2)}{h(-1+h)^2} = \lim\limits_{h \to 0} \frac{2-h}{(-1+h)^2} = 2;$
at $(-1, 1)$: $y = 1 + 2(x-(-1)) \Rightarrow y = 2x + 3$, tangent line



9. $m = \lim\limits_{h \to 0} \frac{(-2+h)^3-(-2)^3}{h} = \lim\limits_{h \to 0} \frac{-8+12h-6h^2+h^3+8}{h}$
$= \lim\limits_{h \to 0} (12-6h+h^2) = 12;$
at $(-2, -8)$: $y = -8 + 12(x-(-2)) \Rightarrow y = 12x + 16$, tangent line



10. $m = \lim_{h \to 0} \frac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^3}}{h} = \lim_{h \to 0} \frac{-8 - (-2+h)^3}{-8h(-2+h)^3}$
$= \lim_{h \to 0} \frac{-(12h - 6h^2 + h^3)}{-8h(-2+h)^3} = \lim_{h \to 0} \frac{12 - 6h + h^2}{8(-2+h)^3}$
$= \frac{12}{8(-8)} = -\frac{3}{16}$;
at $\left(-2, -\frac{1}{8}\right)$: $y = -\frac{1}{8} - \frac{3}{16}(x - (-2))$
$\Rightarrow y = -\frac{3}{16}x - \frac{1}{2}$, tangent line



Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h \to 0} (f(x_0 + h) - f(x_0))/h = \infty$ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h}$$
$$= \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty.$$

T Graph the curves in Exercises 37–46.

a. Where do the graphs appear to have vertical tangents?

b. Confirm your findings in part (a) with limit calculations.

37. $y = x^{2/5}$

38. $y = x^{4/5}$

39. $y = x^{1/5}$

40. $y = x^{3/5}$

41. $y = 4x^{2/5} - 2x$

42. $y = x^{5/3} - 5x^{2/3}$

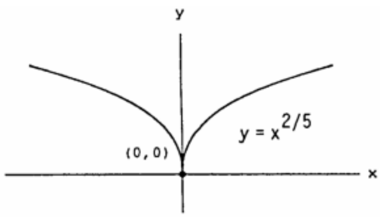
43. $y = x^{2/3} - (x - 1)^{1/3}$

44. $y = x^{1/3} + (x - 1)^{1/3}$

45. $y = \begin{cases} -\sqrt{|x|}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

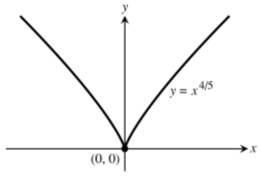
46. $y = \sqrt{|4 - x|}$

37. (a) The graph appears to have a cusp at $x = 0$.



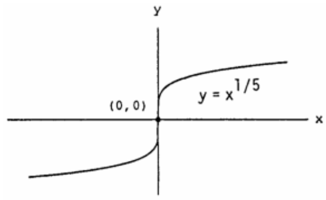
(b) $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/5} - 0}{h} = \lim_{h \to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist
$\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at $x = 0$.

38. (a) The graph appears to have a cusp at $x = 0$.



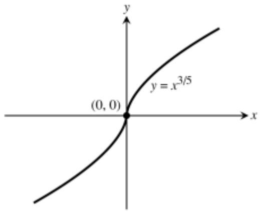
(b) $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \frac{h^{4/5} - 0}{h} = \lim\limits_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim\limits_{h \to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow y = x^{4/5}$ does not have a vertical tangent at $x = 0$.

39. (a) The graph appears to have a vertical tangent at $x = 0$.



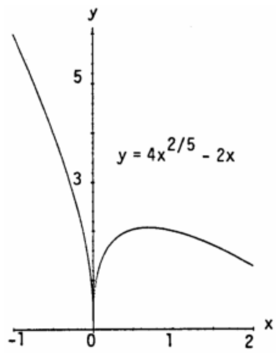
(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{1/5} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at $x = 0$.

40. (a) The graph appears to have a vertical tangent at $x = 0$.



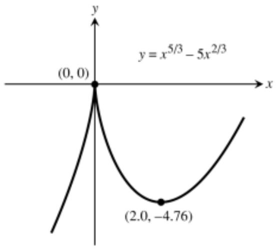
(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{3/5} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

41. (a) The graph appears to have a cusp at $x = 0$.



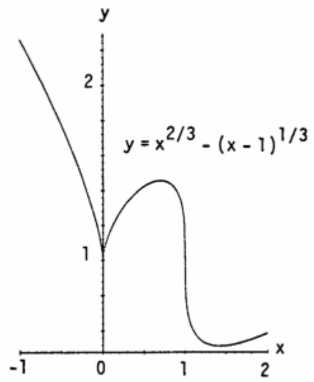
(b) $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \frac{4h^{2/5} - 2h}{h} = \lim\limits_{h \to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim\limits_{h \to 0^+} \frac{4}{h^{3/5}} - 2 = \infty$
$\Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

42. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{5/3} - 5h^{2/3}}{h} = \lim\limits_{h \to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim\limits_{h \to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent at $x = 0$.

43. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



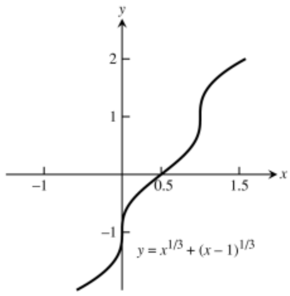
(b) $x = 1$: $\lim_{h \to 0} \frac{(1+h)^{2/3} - (1+h-1)^{1/3} - 1}{h} = \lim_{h \to 0} \frac{(1+h)^{2/3} - h^{1/3} - 1}{h} = -\infty$

$\Rightarrow$ $y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;

$x = 0$: $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{2/3} - (h-1)^{1/3} - (-1)^{1/3}}{h} = \lim_{h \to 0} \left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$

does not exist $\Rightarrow$ $y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at $x = 0$.

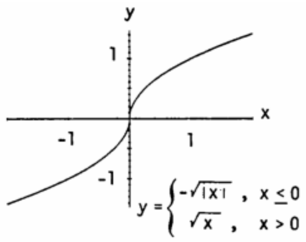
44. (a) The graph appears to have vertical tangents at $x = 0$ and $x = 1$.



(b) $x = 0$: $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} + (h-1)^{1/3} - (-1)^{1/3}}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 0$;

$x = 1$: $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^{1/3} + (1+h-1)^{1/3} - 1}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 1$.

45. (a) The graph appears to have a vertical tangent at $x = 0$.

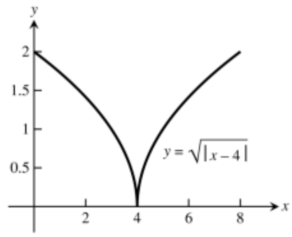


(b) $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{x \to 0^+} \frac{\sqrt{h} - 0}{h} = \lim_{h \to 0} \frac{1}{\sqrt{h}} = \infty$;

$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-\sqrt{|h|} - 0}{h} = \lim_{h \to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim_{h \to 0^-} \frac{1}{\sqrt{|h|}} = \infty$

$\Rightarrow$ y has a vertical tangent at $x = 0$.

46. (a) The graph appears to have a cusp at $x = 4$.



(b) $\lim\limits_{h \to 0^+} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|4-(4+h)|}-0}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty;$

$\lim\limits_{h \to 0^-} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{-1}{\sqrt{|h|}} = -\infty$

$\Rightarrow y = \sqrt{4-x}$ does not have a vertical tangent at $x = 4$.

3-1 The Derivative as a Function:

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

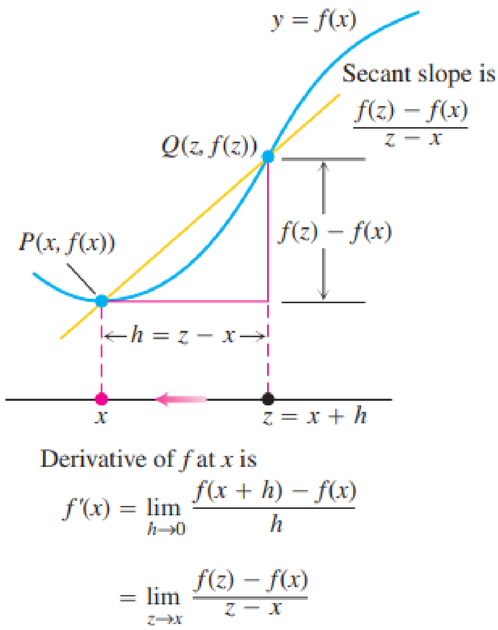


FIGURE 3.4 Two forms for the difference quotient.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

3-3 differentiation Rules:

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx} x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is any real number, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Ex.

find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

5. $y = \frac{4x^3}{3} - x + 2e^x$

7. $w = 3z^{-2} - \frac{1}{z}$

28. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$

30. $y = \frac{x^2 + 3e^x}{2e^x - x}$

32. $w = re^{-r}$

34. $y = x^{-3/5} + \pi^{3/2}$

13. $y = (3 - x^2)(x^3 - x + 1)$

Solution:

1. $y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$

2. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

5. $y = \frac{4}{3}x^3 - x \Rightarrow \frac{dy}{dx} = 4x^2 - 1 \Rightarrow \frac{d^2y}{dx^2} = 8x$

7. $w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$

28. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2} = \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

30. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

31. $y = (x-1)(x^2+3x-5) = x^3 + 2x^2 - 8x + 5 \Rightarrow y' = 3x^2 + 4x - 8 \Rightarrow y'' = 6x + 4 \Rightarrow y''' = 6 \Rightarrow y^{(n)} = 0$ for all $n \geq 4$

13. (a) $y = (3-x^2)(x^3-x+1) \Rightarrow y' = (3-x^2)\cdot\frac{d}{dx}(x^3-x+1) + (x^3-x+1)\cdot\frac{d}{dx}(3-x^2)$
$= (3-x^2)(3x^2-1) + (x^3-x+1)(-2x) = -5x^4 + 12x^2 - 2x - 3$

(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

32. $y = (4x^3+3x)(2-x) = -4x^4 + 8x^3 - 3x^2 + 6x \Rightarrow y' = -16x^3 + 24x^2 - 6x + 6 \Rightarrow y'' = -48x^2 + 48x - 6$
$\Rightarrow y''' = -96x + 48 \Rightarrow y^{(4)} = -96 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

34. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

\>

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x$$

(b) $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)\sin x \qquad \text{Product Rule}$$

$$= e^x \cos x + e^x \sin x$$

$$= e^x(\cos x + \sin x)$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

■

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

EXAMPLE 2 We find derivatives of the cosine function in combinations with other functions.

(a) $y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$

$$= 5e^x - \sin x$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

\>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 5 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

EXAMPLE 6 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x && \text{Derivative rule for secant function} \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Derivative Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) && \text{Derivative rules} \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

■

Ex.

find dy/dx.

5. $y = \csc x - 4\sqrt{x} + 7$

6. $y = x^2 \cot x - \dfrac{1}{x^2}$

7. $f(x) = \sin x \tan x$

8. $g(x) = \csc x \cot x$

9. $y = (\sec x + \tan x)(\sec x - \tan x)$

10. $y = (\sin x + \cos x) \sec x$

11. $y = \dfrac{\cot x}{1 + \cot x}$

12. $y = \dfrac{\cos x}{1 + \sin x}$

34. Find $y^{(4)} = d^4y/dx^4$ if

a. $y = -2 \sin x$.

b. $y = 9 \cos x$.

5. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$

6. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2 \csc^2 x + 2x \cot x + \frac{2}{x^3}$

7. $f(x) = \sin x \tan x \Rightarrow f'(x) = \sin x \sec^2 x + \cos x \tan x = \sin x \sec^2 x + \cos x \frac{\sin x}{\cos x} = \sin x(\sec^2 x + 1)$

8. $g(x) = \csc x \cot x \Rightarrow g'(x) = \csc x(-\csc^2 x) + (-\csc x \cot x)\cot x = -\csc^3 x - \csc x \cot^2 x = -\csc x(\csc^2 x + \cot^2 x)$

9. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$
$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

10. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

11. $y = \frac{\cot x}{1 + \cot x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2} = \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} = \frac{-\csc^2 x}{(1 + \cot x)^2}$

12. $y = \frac{\cos x}{1 + \sin x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} = \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} = \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$

34. (a) $y = -2 \sin x \Rightarrow y' = -2 \cos x \Rightarrow y'' = -2(-\sin x) = 2 \sin x \Rightarrow y''' = 2 \cos x \Rightarrow y^{(4)} = -2 \sin x$
(b) $y = 9 \cos x \Rightarrow y' = -9 \sin x \Rightarrow y'' = -9 \cos x \Rightarrow y''' = -9(-\sin x) = 9 \sin x \Rightarrow y^{(4)} = 9 \cos x$