Functions

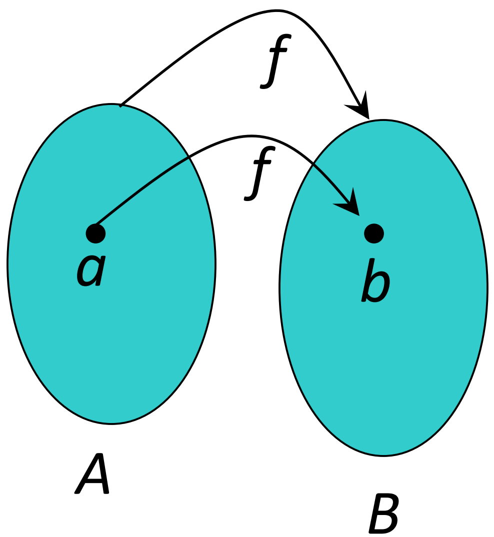
Review of Functions

Mathematics is a language with an alphabet, a vocabulary, and many rules. Everywhere around us we see relationships among quantities, or variables. For example, the consumer price index changes in time and the temperature of the ocean varies with latitude. These relationships can often be expressed by mathematical objects called function.

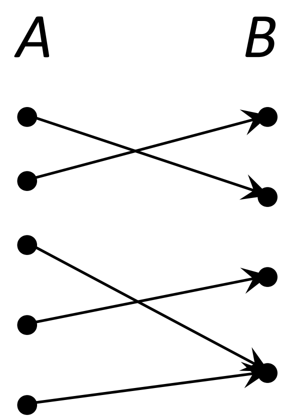
DEFINITION:
A function *from a set D to set R is a rule that assigns a unique element in R to each element in D .*

Graphical Representations

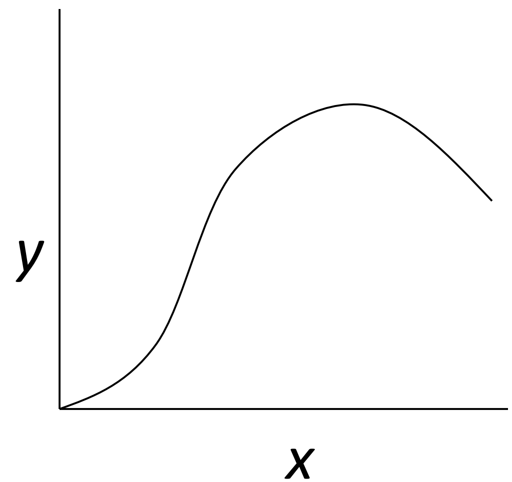
- Functions can be represented graphically in several ways:



Like Venn diagrams

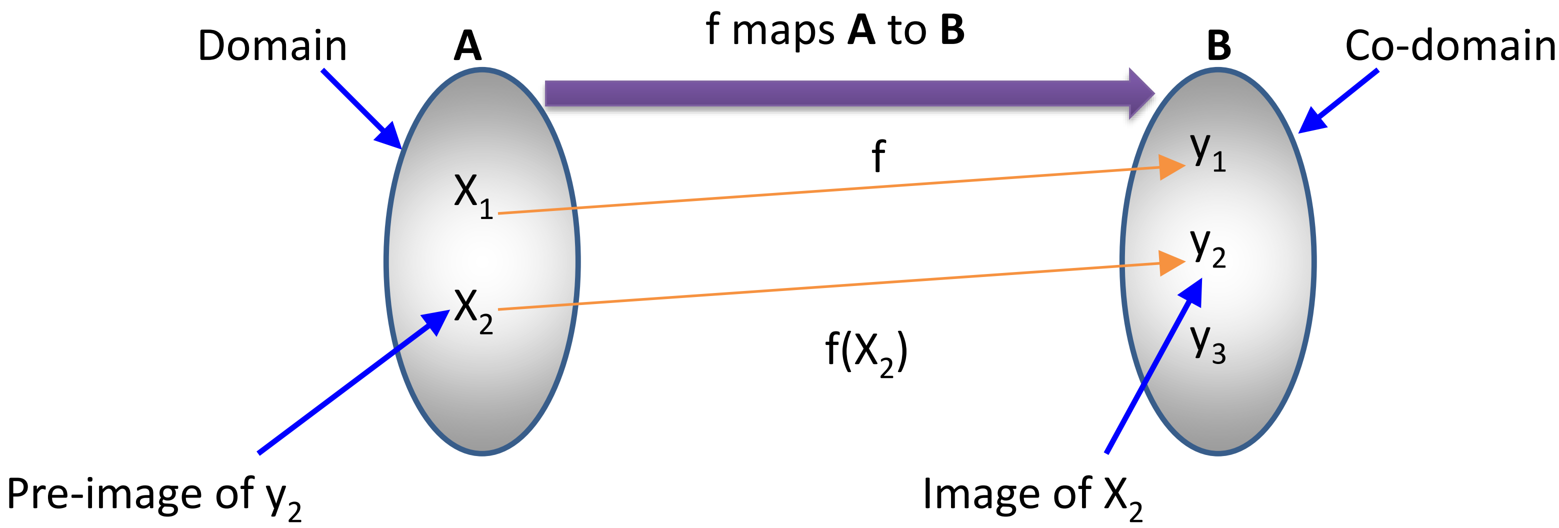


Graph



Plot

Function terminology



- Domain (منطلق) is the set A of “input” or argument values for f.
- Codomain (مستقر) is the set B into which all of the output of f.
- Image is the subset of function’s codomain.
- Preimage is the inverse image of a subset.

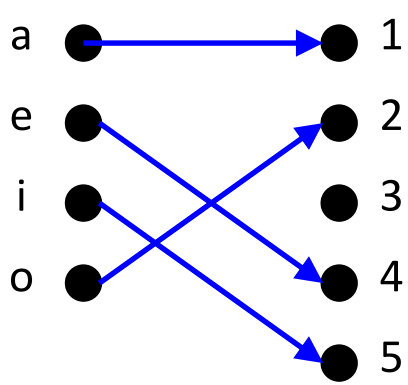
One-to-one functions (Injection) دالة متباينة

- A function is one-to-one if every element of the function's codomain is the image of at most one element of its domian.

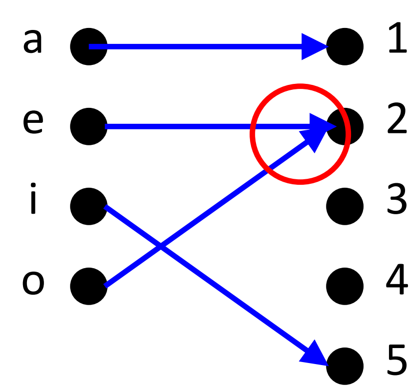
هي دالة تبقى بها العناصر متباينة لا تقترن العناصر المتباينية من مجالها بنفس العنصر من مجالها المقابل بمعنى أن كل عنصر من مجالها المقابل مقترن بعنصر من مجالها واحد على الأكثر

$$\forall\, x, y \in A, f(x) = f(y) \rightarrow x = y$$

$$\forall\, x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$$



A one-to-one function



A function that is not one-to-one

$$Let\ f: R \to R, R\ is\ a\ real\ number, f(x) = x^2$$

$$\forall x \in R, f(x) = x^2, f(-x) = x^2$$

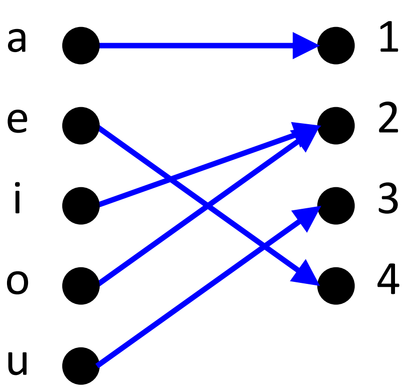
$$f(1) = 1, f(-1) = 1$$

$$f(2) = 4\ , f(-2) = 4$$

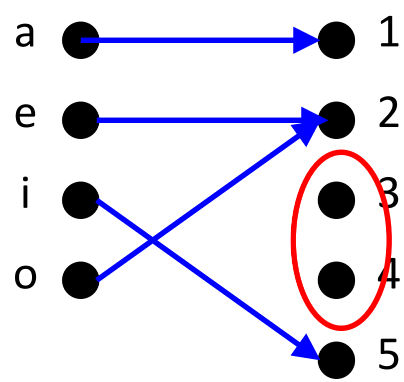
دالة غير متباينة

Onto functions (Surjection) دالة شاملة

- The function f:A→B is called surjection (or onto) function if f(A)=B, هي دالة يكون مداها مساويا للمجال المقابل.
  - Meaning the range is equal to the codomain



An onto function

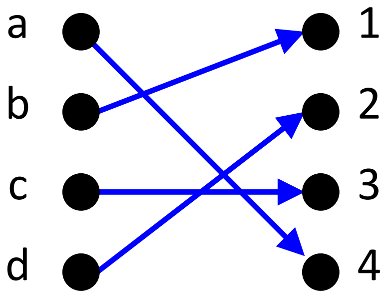


A function that is not onto

اذا كان كل عنصر من مستقرها (codomain) هو صور لعنصر واحد على الاقل من المنطلق (domain)

Bijection (One-to-one correspondence)
دالة تقابلية

- Consider a function that is both one-to-one and onto:



- Such a function is a one-to-one correspondence, or a bijection

تكون الدالة تقابلية اذا كانت متباينة وشاملة.

Identity functions (حيادية) دالة متطابقة

- The function f:A→B is called identity function if

$$\forall\, x \in A, f(x) = x$$

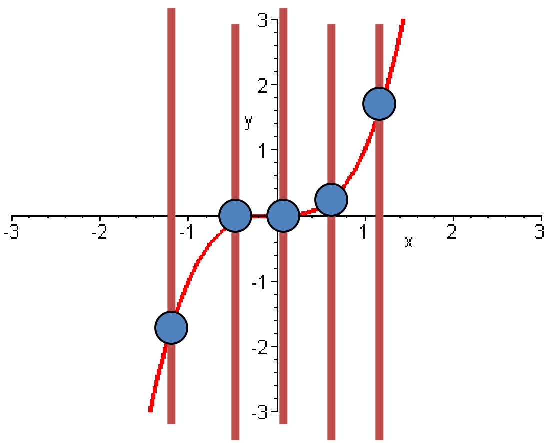
- f(x) = 1*x
- f(x) = x + 0
- The domain and the range must be the same set

الدالة الحيادية او الذاتية هي دالة متباينة وشاملة وتقابلية.

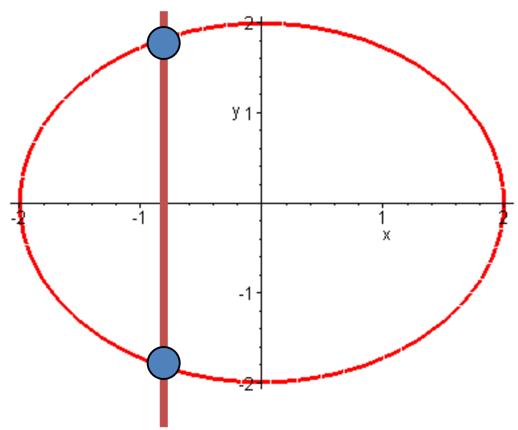
Vertical Line Test

To determine by the graph if an equation is a function, we have the vertical line test.

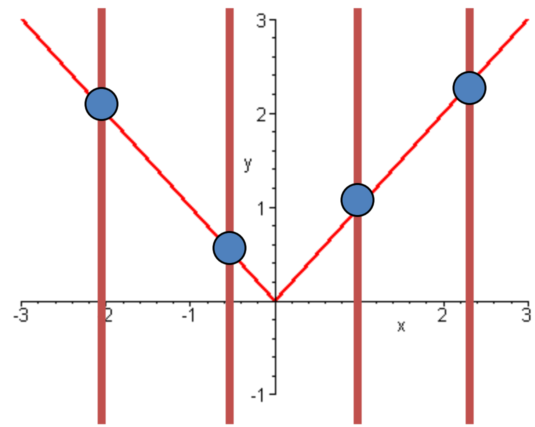
If a vertical line intersects the graph of an equation more than one time, the equation graphed is NOT a function.



This is a function



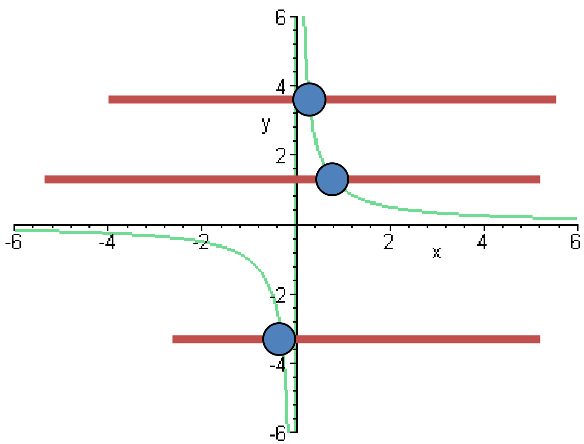
This is NOT a function



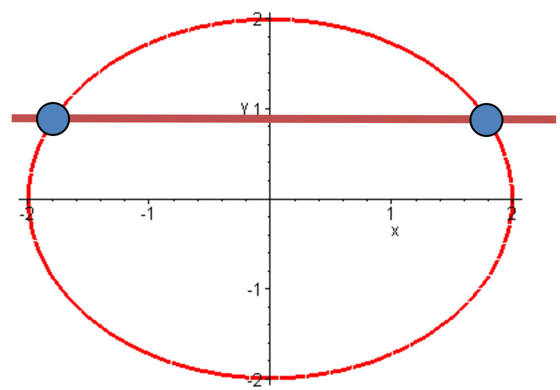
This is a function

Horizontal Line Test

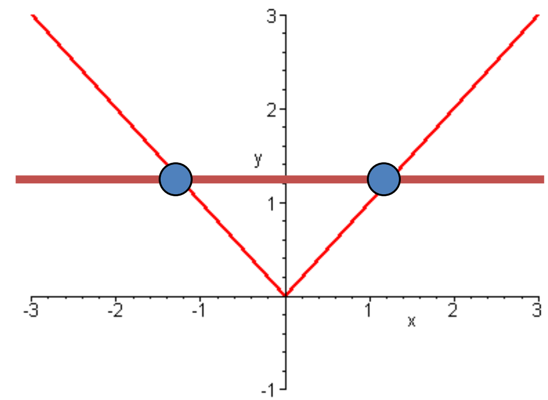
If a horizontal line intersects the graph of an equation more than one time, the equation graphed is NOT a one-to-one function and will NOT have an inverse function.



This is a one-to-one function



This is NOT a one-to-one function



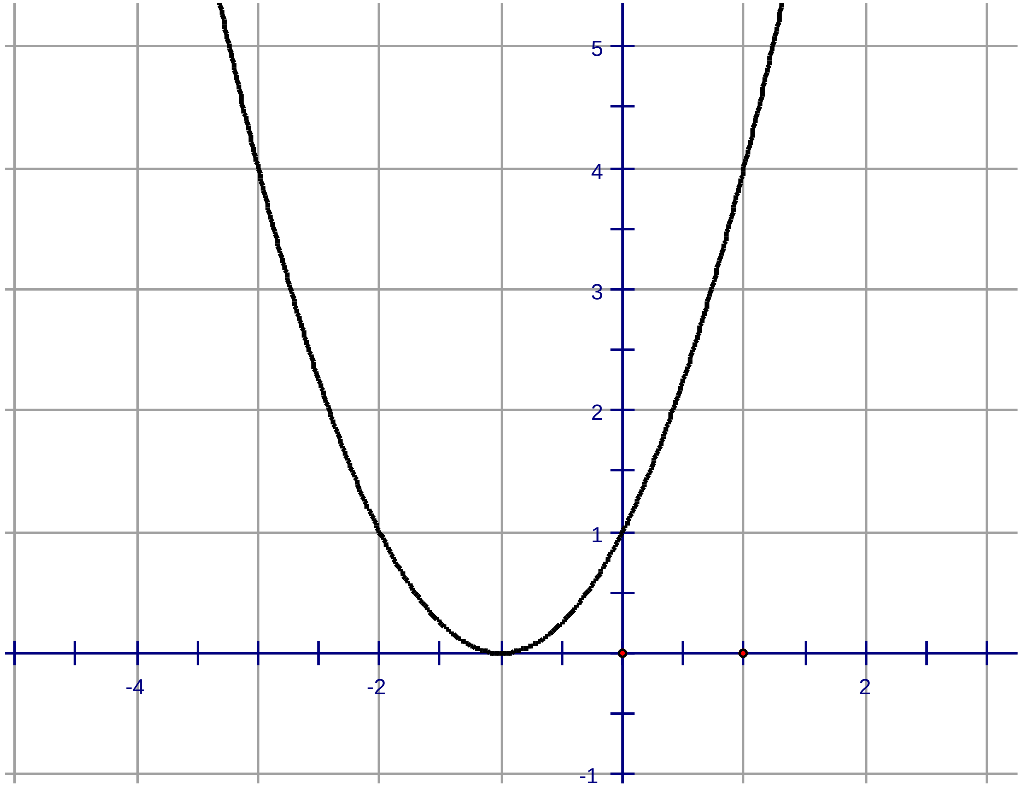
This is NOT a one-to-one function

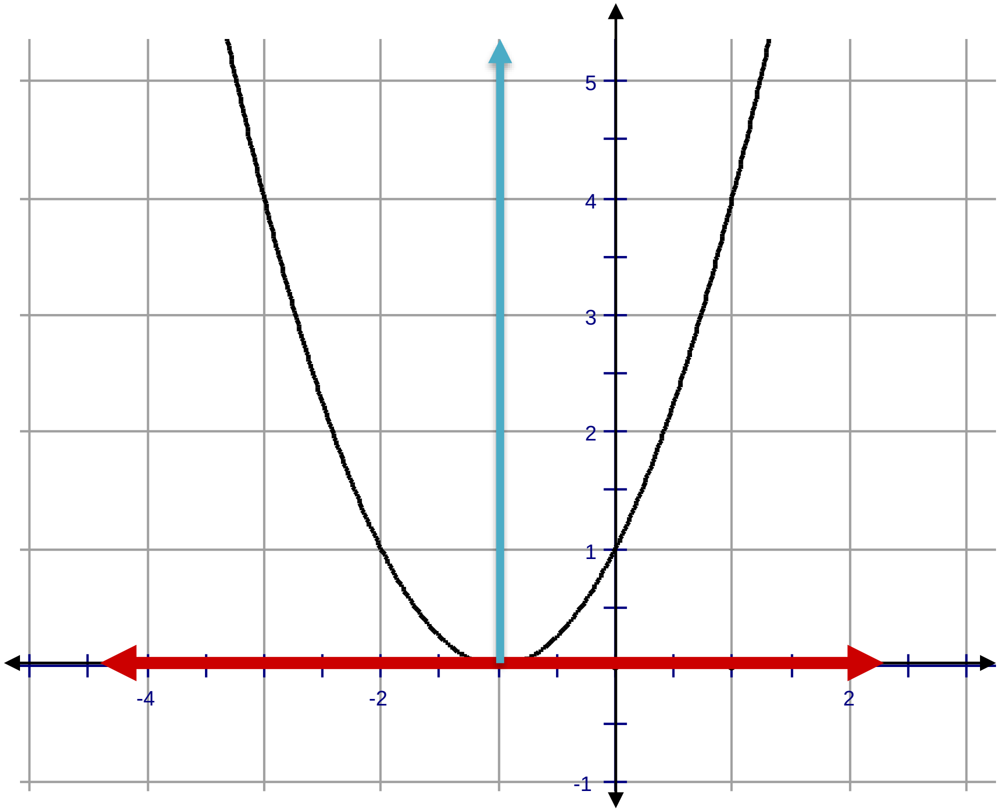
Domain: In a set of ordered pairs, (x, y), the *domain* is the set of all x-coordinates.

Range: In a set of ordered pairs, (x, y), the *range* is the set of all y-coordinates.

The set of ordered pairs may be an <u>infinite number of points</u>, described by a graph.

Given the following graph, find the domain and range.





Domain:{all real numbers}

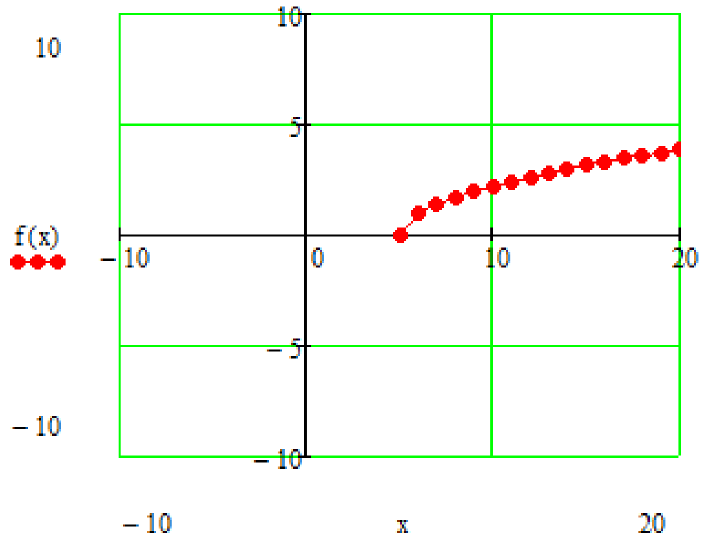
Range:**{y:y≥0}**

The set of ordered pairs may be an infinite number of points, described by an algebraic expression.

Example:

Given the following function, find the domain and range.

$$f(x) = \sqrt{x-5}$$



Domain: {x: x≥5}

Range: {y: y≥0}

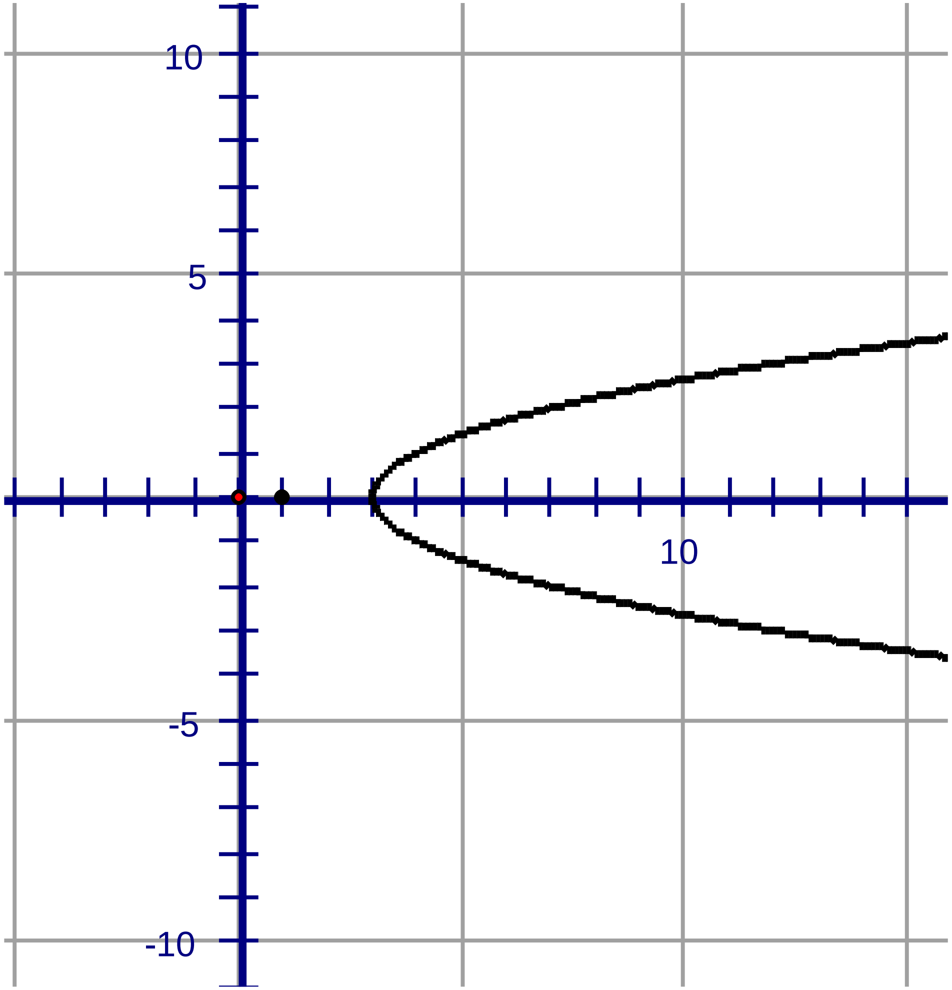
Practice: Find the domain and range of the following sets of ordered pairs.

1. {(3,7),(-3,7),(7,-2),(-8,-5),(0,-1)}

Domain:{3,-3,7,-8,0}

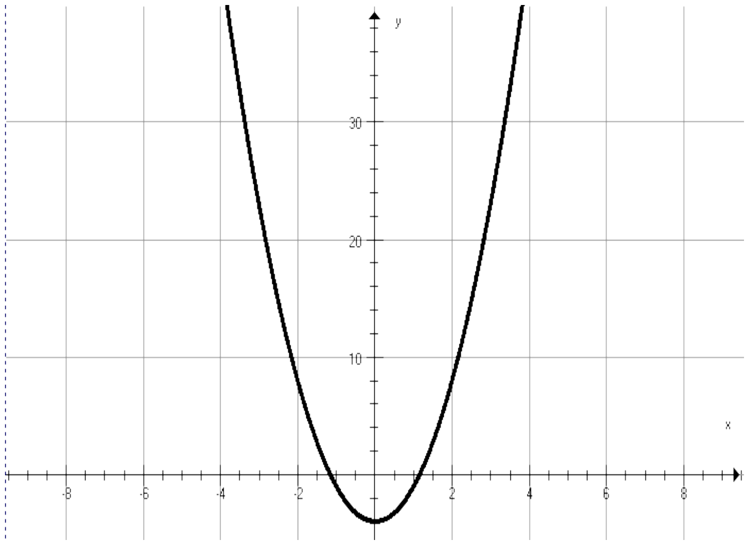
Range:{7,-2,-5,-1}

2.



Domain={x:x ≥ 3} Range:{all reals}

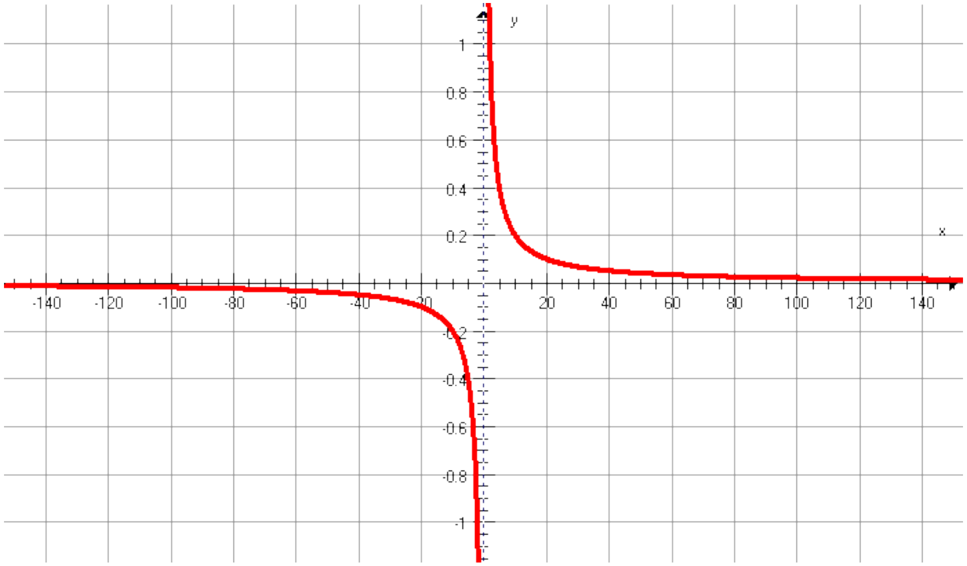
3. $f(x) = 3x^2 - 4$



Domain={all reals}

Range:{y:y≥-4}

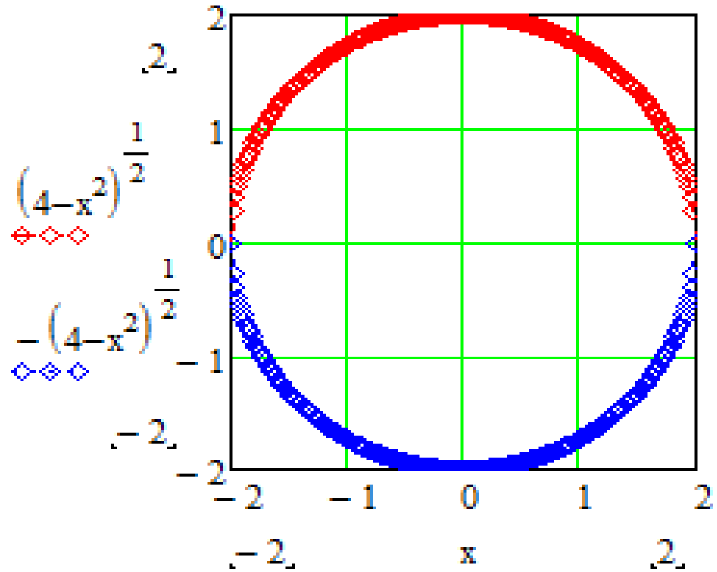
4. $f(x) = \frac{2}{x}$



Domain={x:x≠0}

Range:{y:y≠0}

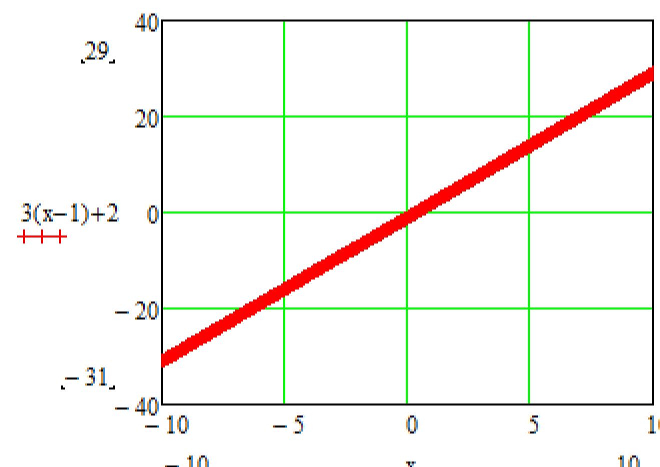
5. $x^2 + y^2 = 4$



Domain={x: -2≤x≤2}

Range:{*y:* -2≤y≤2}

6. $f(x) = 3(x-1)+2$



Domain={all reals}

Range:{all reals}

Steps to find range of function

- To find the range of a function f described by formula, where the domain is taken to be the natural domain,

1) Put $y=f(x)$;
2) Solve x in terms of y;
3) The range of f is the set of all numbers y such that x can be solved.

Example:

Find the range of function $f(x) = x^2 + 2$

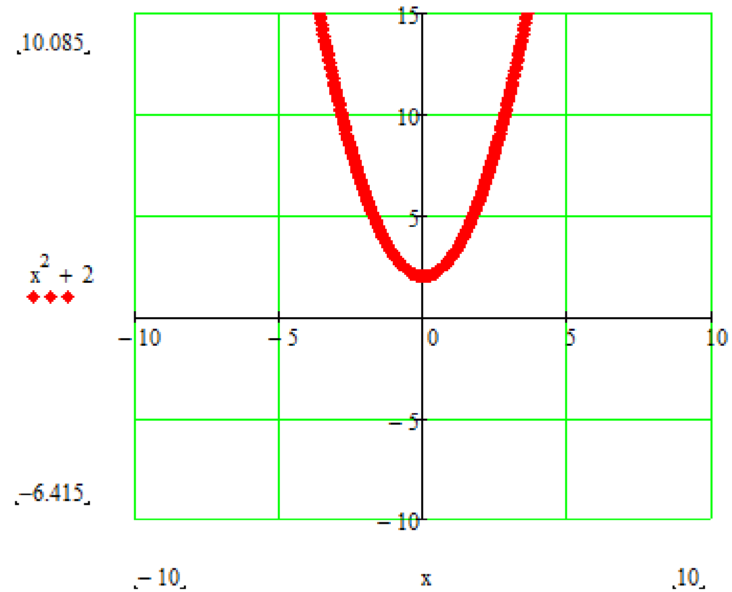
Solve:

$$f(x) = x^2 + 2$$

$$y = f(x) = x^2 + 2$$

$$x^2 = y - 2$$

$$x = \pm\sqrt{y - 2}$$



Note that x can be solved iff $y - 2 \geq 0$

The range of f is $\{y \in \mathbb{R}: y - 2 \geq 0\}$

$$= \{y \in \mathbb{R}: y \geq 2\} = [2, \infty]$$

Example:

Find the domain and the range of function $g(t)=\sqrt{4-t^2}$

Solve:

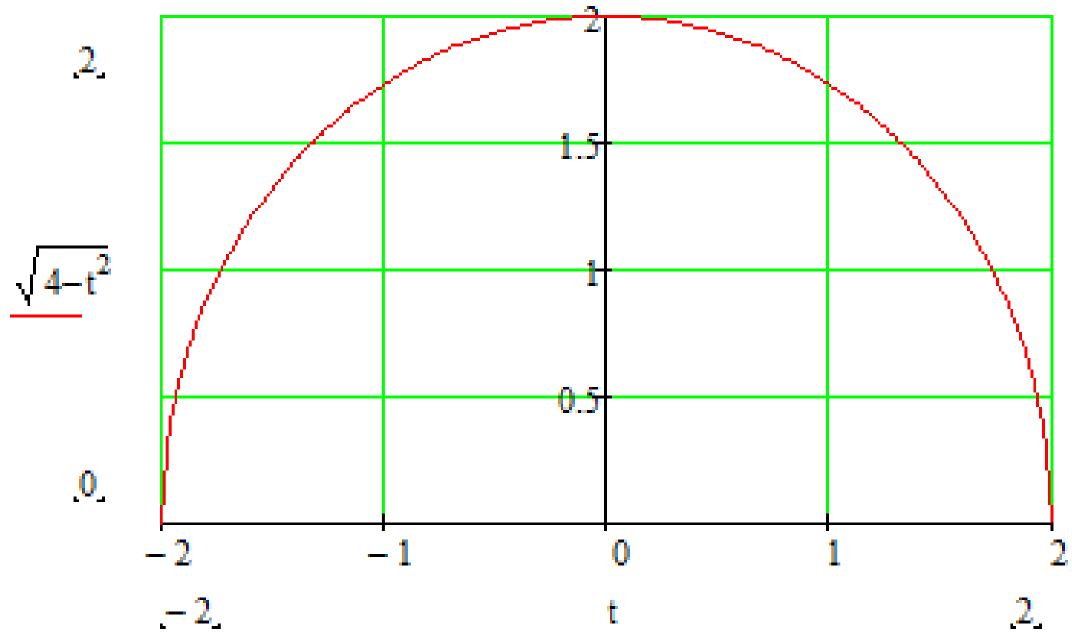
$$g(t)=\sqrt{4-t^2}$$

$$z=g(t)=\sqrt{4-t^2}$$

$$t^2=4-z^2$$

$$t=\pm\sqrt{4-z^2}$$

The domain of g is [-2,2]



When t=0, z reaches its maximum value of $g(0)=\sqrt{4}=2$
When t=±2, z attains its minimum value of g(±2)=0. Therefore, the range of g is [0,2]

Example:

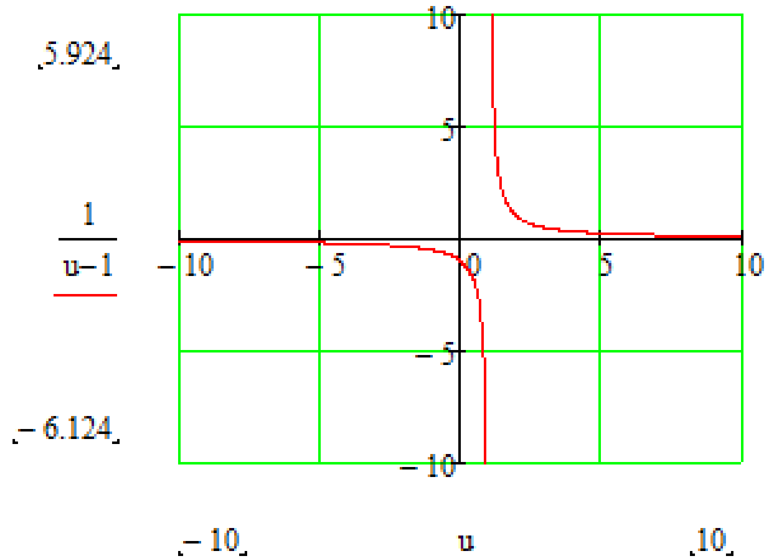
Find the domain and the range of function $h(u)=\dfrac{1}{u-1}$

Solve:

$$h(u)=\frac{1}{u-1}$$

$$w=h(u)=\frac{1}{u-1}$$

$$u=\frac{1}{w}+1$$



The function h is undefined at u=1, so its domain is {u: u≠1} and the graph does not have a point corresponding to u=1

We see that w takes on all values except 0; therefore, the range is {w: w≠0}

Example:

Find the domain and the range of $f(x) = 5 - x^2$

Solve:

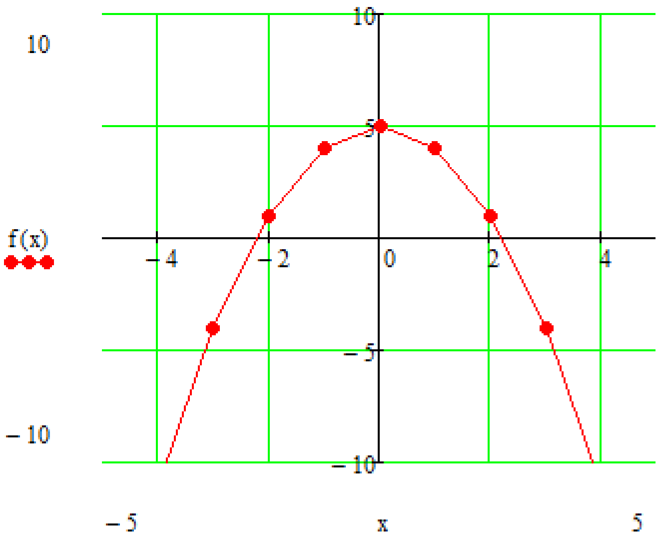
The domain is $D_f = R$

$$y = 5 - x^2$$

$$x = \sqrt{5 - y}$$

The range is

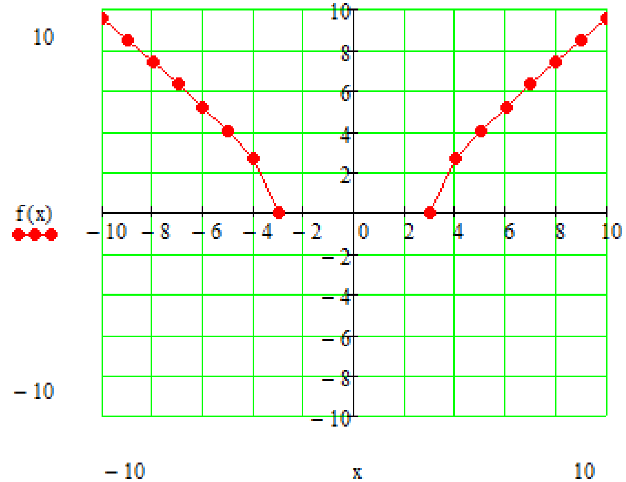
$$R_f =]-\infty, 5]$$



Example:

Find the domain and the range of $f(x)=\sqrt{x^2-9}$

Solve:



To find the domain :

$x^2 - 9 \geq 0$

$x^2 \geq 9$

$\sqrt{x^2} \geq \sqrt{9} = 3$

$|x| \geq 3$

$x \geq 3 \text{ or } x \leq -3$

Thus

$D_f = \{x \mid x \in R; x \geq 3 \text{ or } x \leq -3\} =]-\infty,-3] \cup [3,\infty[$

To find the range :

$y = \sqrt{x^2 - 9}$

$x = \sqrt{y^2 + 9}$

so the range is R but from the function $y = \sqrt{x^2 - 9}$

y does not take the (-ive) value so $R_f = R^+ \cup \{0\}$.

Remark: Let us denote the domain of any function *f* by *D(f)* so

$$1.\ D(f+g)=D(f-g)=D(f.g)=D(f)\bigcap D(g).$$

$$2.\ D(f/g)=D(f)\bigcap D(g)-\{x \mid g(0)=0\}.$$

Example:

Find D_f of $f_1(x)=\sqrt{(x+1)(4-x^2)}$, and $f_2(x)=\sqrt{(x+1)/(4-x^2)}$

Solve:

$$\text{Let } h(x)=\sqrt{(x+1)} \text{ and } g(x)=\sqrt{(4-x^2)}$$

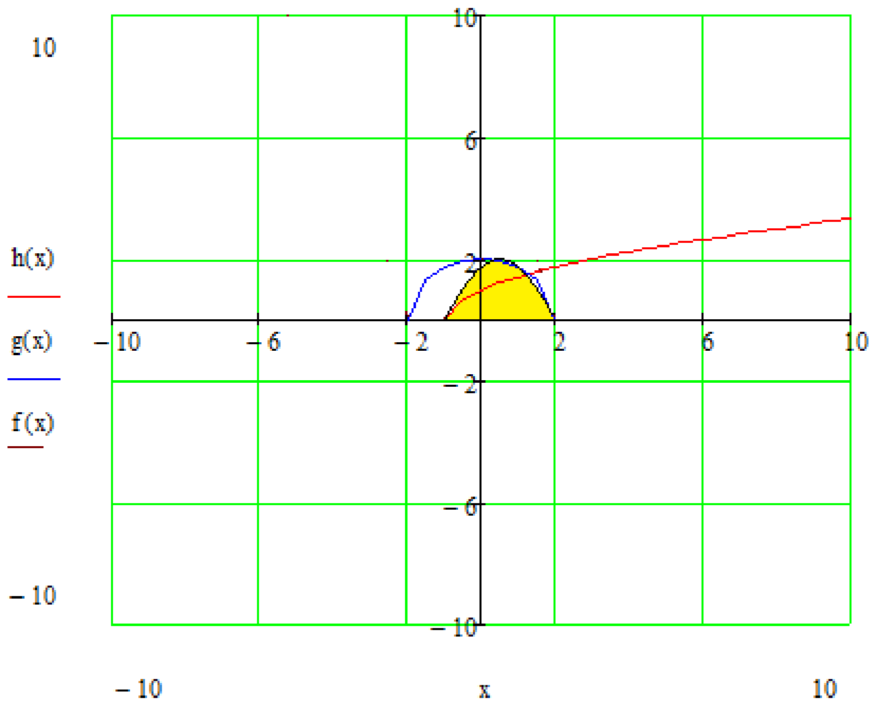
$$D(h)=D_h=[-1,\infty[\text{ and } D(g)=D_g=[-2,2].$$

Since by remark before we can write:

$$D(f_1)=D(h.g)=D(h)\cap D(g)=[-1,\infty[\cap[-2,2]=[-1,2]$$

$$D(f_2) = D(h/g) = D(h) \bigcap D(g) - \{x \mid g(0) = 0\}$$

$$= [-1,2] - \left\{x : \sqrt{(4 - x^2)} = 0\right\} = [-1,2] - \{2,-2\} = [-1,2[.$$



Example:

Find the domain and the range of $f(x)=\dfrac{x^2+2}{x^2-2}$.

Solve:

Let $h(x)=x^2+2$ and $g(x)=x^2-2$

So $D(h)=R$ and $D(g)=R$

Thus :

$D(f)=D(h/g)=D(h)\bigcap D(g)-\{x\mid g(0)=0\}$

$=R\bigcap R-\{x\mid x^2-2\}=0$

$=R-\{\sqrt{2},-\sqrt{2}\}$

To find the range:

$$y = \frac{x^2 + 2}{x^2 - 2}$$

$$y(x^2 - 2) = x^2 + 2$$

$$yx^2 - 2y = x^2 + 2$$

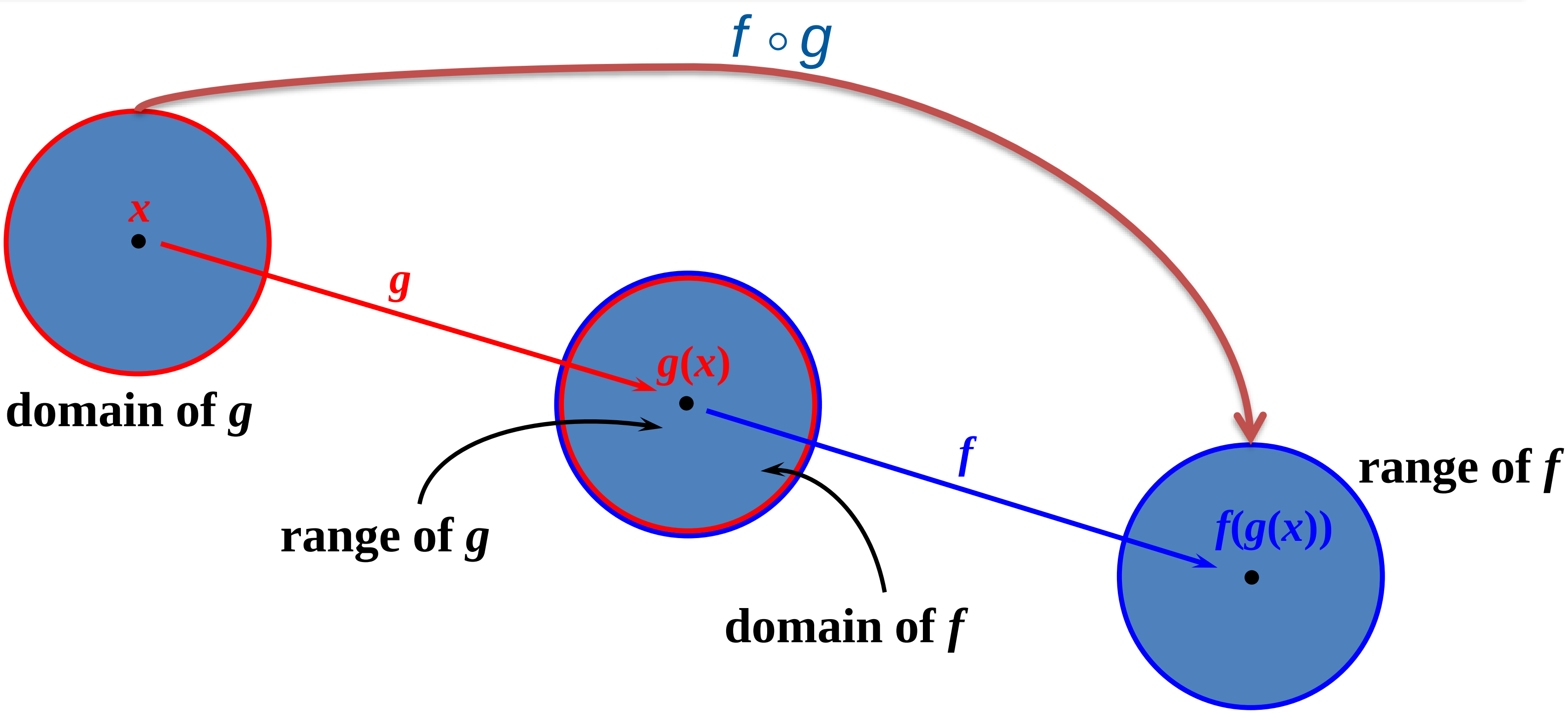
$$x^2(y - 1) = 2y + 2$$

$$x = \sqrt{\frac{2y + 2}{y - 1}}$$

$$\left.\begin{cases} 2y + 2 \geq 0 \rightarrow y \geq -1 \rightarrow \left]-1, \infty\right[\\ y - 1 > 0 \rightarrow \left]1, \infty\right[\end{cases}\right\} \rightarrow R_f = \left]1, \infty\right[$$

Function Composition

- Let *f* and *g* be two functions, the composite function ($f \bigcirc g$) is defined by: $(f \bigcirc g)(x)=f(g(x))$
- The domain of $f \bigcirc g$ consists of the numbers *x* in the domain of g for which *g(x)* lies in the domain of *f*

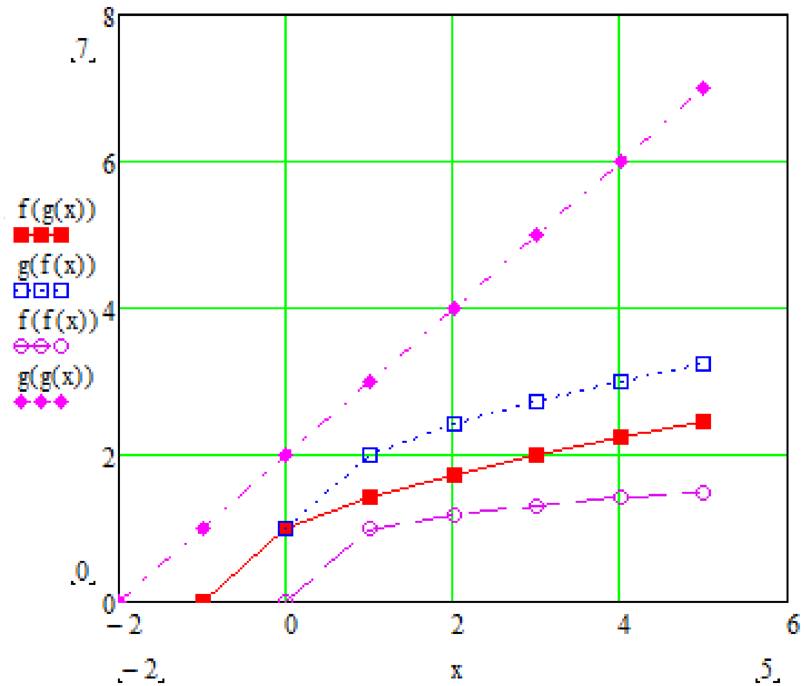


Example:

Finding Formulas for composites

$$f(x) = \sqrt{x} \;\; and \; g(x) = x + 1$$

$$a - (f \circ g)(x), b - (g \circ f)(x),$$
$$c - (f \circ f)(x), d - (g \circ g)(x)$$



Solution

$$a - (f \circ g)(x) = f\left(g(x) = \sqrt{g(x)}\right) = \sqrt{x+1}, [-1, \infty)$$

$$b - (g \circ f)(x) = g\big(f(x)\big) = f(x) + 1 = \sqrt{x} + 1, [0, \infty)$$

Remark: Let f:A to B and g:B to A be two real functions then:

1. (i) if f and g are injection (1-1) function
   then $(f \circ g)$ is injection.

   (ii) if f and g are surjection (onto) function
   then $(f \circ g)$ is surjection.

   (iii) if f and g are bijection function
   then $(f \circ g)$ is bijection.

2. $D(f \circ g) = \{x \mid x \in D(g) \text{ and } g(x) \in D(f)\}$.

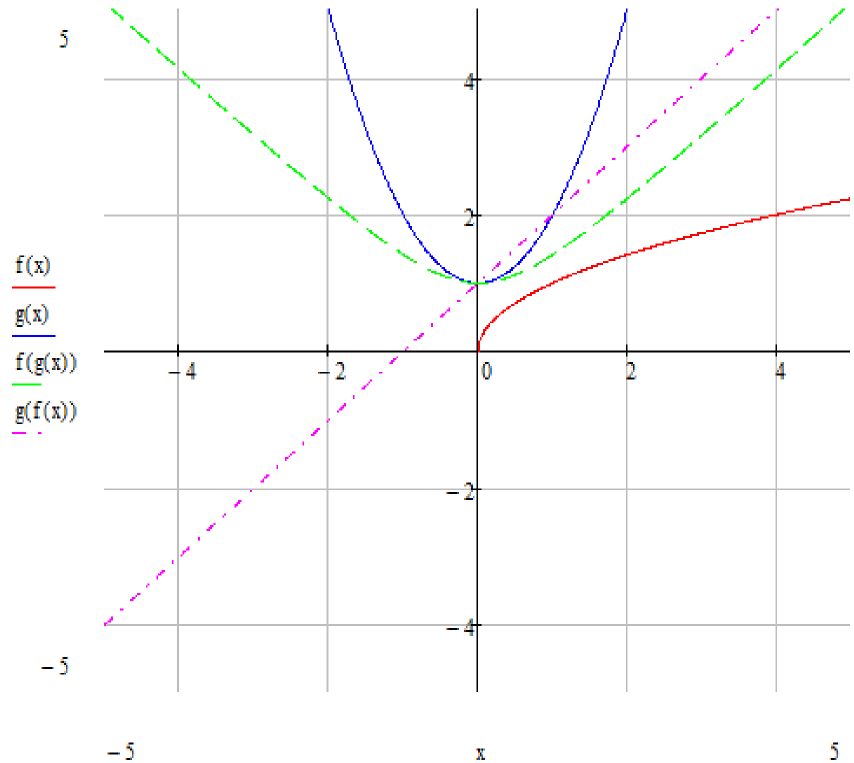
Example:

$$Let \ f(x) = \sqrt{x} \text{ and } g(x) = x^2 + 1.$$

Solution

$$(fog)(x) = \sqrt{x^2 + 1}$$

$$(gof)(x) = (\sqrt{x})^2 + 1 = x + 1$$



Now $D(f) = R^+ \cup \{0\}$ and $D(g) = R$, so :

$$D(fog) = \{x \mid x \in R \text{ and } g(x) = x^2 + 1 \in D(f)\} \text{but } x^2 + 1 > 0$$

and $x^2 + 1 \in D(f)$ for all real numbers. So that $D(fog) = R.$

Odd and Even Function: The real function

$f : X \to R$ is odd fuction if $\forall x \in X$, $f(-x) = -f(x)$;
and even function if $\forall x \in X$, $f(-x) = f(x)$.

Example:

$f(x) = x^2 + 1$ it is even function since

$$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$$

$f(x) = x^5 - x^3 + x$ it is odd function since

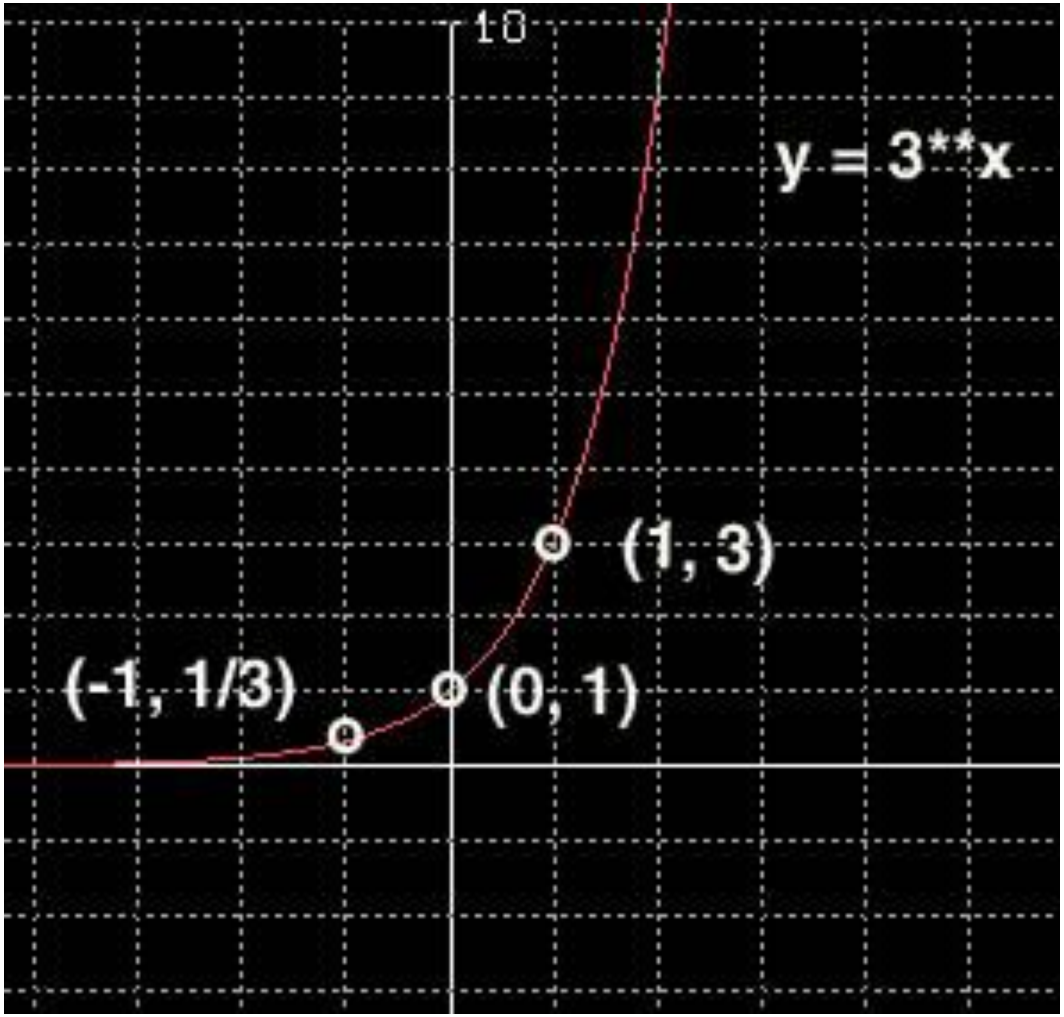
$$f(-x) = (-x)^5 - (-x)^3 + (-x) = -(x^5 - x^3 + x) = -f(x)$$

General Form of Exponential Function

$y = b^x$ where $b > 1$

- Domain: All reals
- Range:

  $y > 0$
- x-intercept: None
- y-intercept: (0, 1)



Properties of Exponents

Let a and b be real numbers and m and n be integers. Then the following properties of exponents:

1. $a^m a^n = a^{m+n}$

2. $(a^m)^n = a^{mn}$

3. $(ab)^m = a^m b^m$

4. $\dfrac{a^m}{a^n} = a^{m-n}, \; a \neq 0$

5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}, \; b \neq 0$

6. $a^{-m} = \dfrac{1}{a^m}, \; a \neq 0$

7. $a^{\frac{1}{n}} = \sqrt[n]{a}$

8. $a^0 = 1, \; a \neq 0$

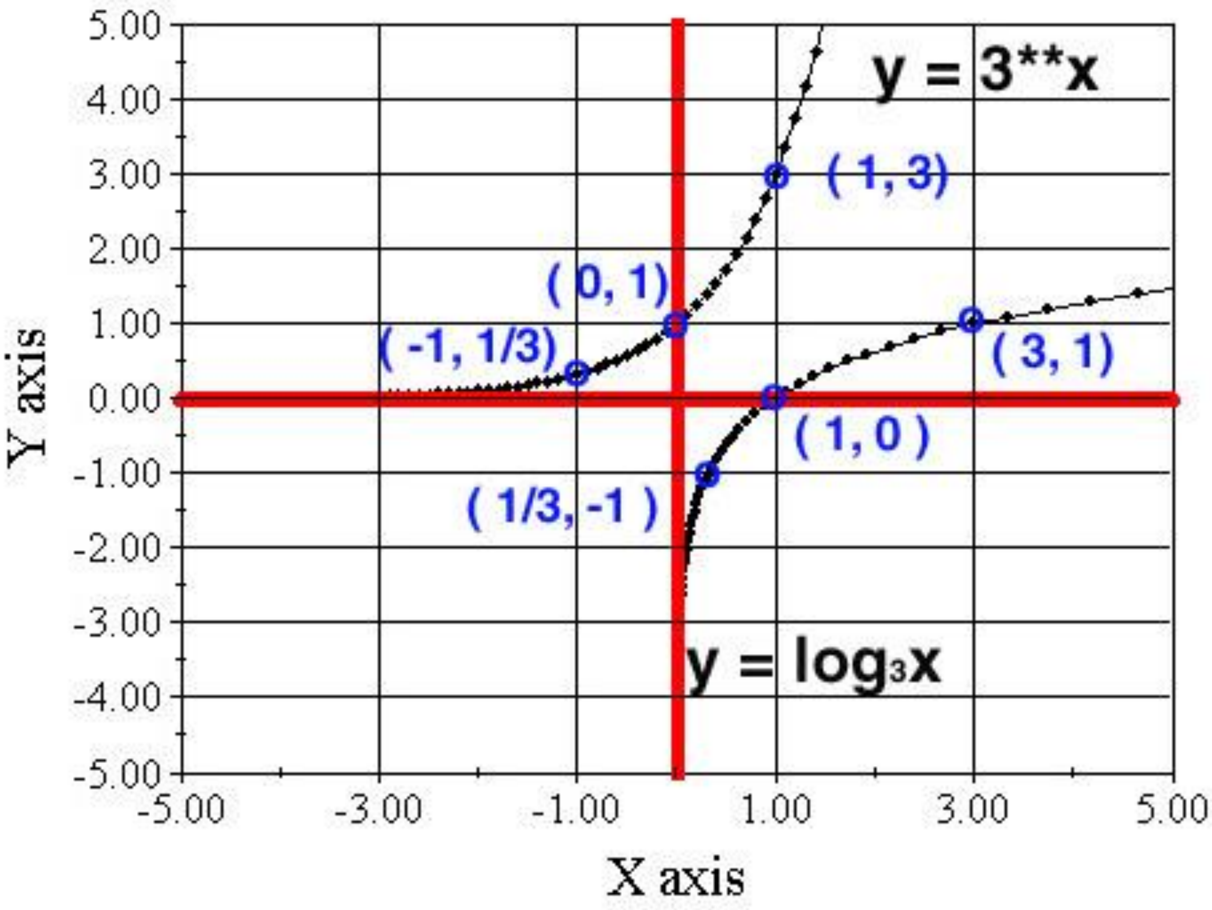
9. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Relationships of Exponential ($y = b^x$) & Logarithmic ($y = \log_b x$) Functions

- $y = b^x$
- Domain: All Reals
- Range: $y > 0$
- x-intercept: None
- y-intercept: (0, 1)

- $y = \log_b x$ is the inverse of $y = b^x$
- Domain: $x > 0$
- Range: All Reals
- x-intercept: (1, 0)
- y-intercept: None

Relationships of Exponential ($y = b^x$) & Logarithmic ($y = \log_b x$) Functions

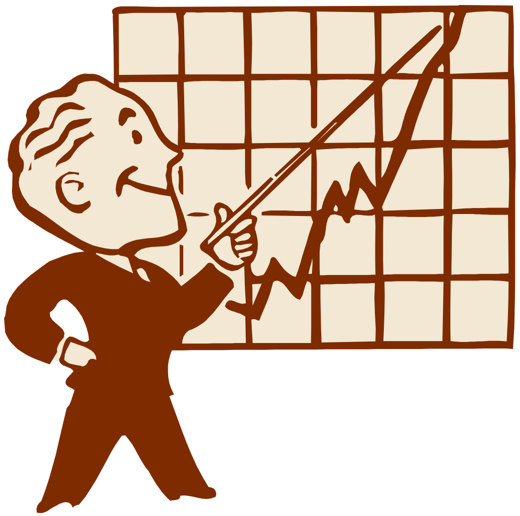


Converting between Exponents & Logarithms

- $\text{BASE}^{\text{EXPONENT}} = \text{POWER}$
- $4^2 = 16$
- 4 is the base. 2 is the exponent.
  16 is the power.
- As a logarithm, $\log_{\text{BASE}}\text{POWER}=\text{EXPONENT}$
- $\log_4 16 = 2$

Logarithmic Abbreviations

- $\log_{10} x = \log\ x$ (Common log)
- $\log_e x = \ln x$ (Natural log)
- $e = 2.71828...$



Properties of Logarithms

- **$\log_b(XY) = \log_b X + \log_b Y$**

  Ex: $\log_4(15) = \log_4 5 + \log_4 3$

- **$\log_b(X/Y) = \log_b X - \log_b Y$**

  Ex: $\log_3(50/2) = \log_3 50 - \log_3 2$

- **$\log_b X^n = n \log_b X$**

  Ex: $\log_7 10^3 = 3 \log_7 10$

- **$\log_b(1/X) = \log_b X^{-1} = -1 \log_b X = -\log_b X$**

  $\log_{11}(1/8) = \log_{11} 8^{-1} = -1 \log_{11} 8 = -\log_{11} 8$

Properties of Logarithms (Shortcuts)

- $\log_b 1 = 0$ (because $b^0 = 1$)
- $\log_b b = 1$ (because $b^1 = b$)
- $\log_b b^n = n$ (because $b^n = b^n$)
- $b^{\log_b X} = X$ (because $\log_b X = \log_b X$)

Examples of Logarithms

- Simplify log 7 + log 4 – log 2 =

$$\log \frac{7*4}{2} = \log 14$$

- Simplify ln e^2 =

$$2 \ln e = 2 \log_e e = 2 * 1 = 2$$

- Simplify $e^{4 \ln 3 - 3 \ln 4}$ =

$$e^{\ln 3^4 - \ln 4^3} = e^{\ln 81/64} = e^{\log_e 81/64} = 81/64$$

Change-of-Base Formula

Change of base formula can be represented as follows. Here the base of the given logarithm is also changed to a logarithm with a new base. The basic logarithm with a base is transformed to two logarithms with a new and same base. The change of base formula is:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b a \cdot \log_c b = \log_c a$$

Example: Evaluate the value of $\log_{64} 8$ using the change of base formula.

We will apply the change of base formula (by changing the base to 10). Note that $\log_{10}$ is same as log.

$$\log_{64} 8 = \frac{\log 8}{\log 64} = \frac{\log 8}{\log 8^2} = \frac{\log 8}{2 \log 8} = \frac{1}{2}$$

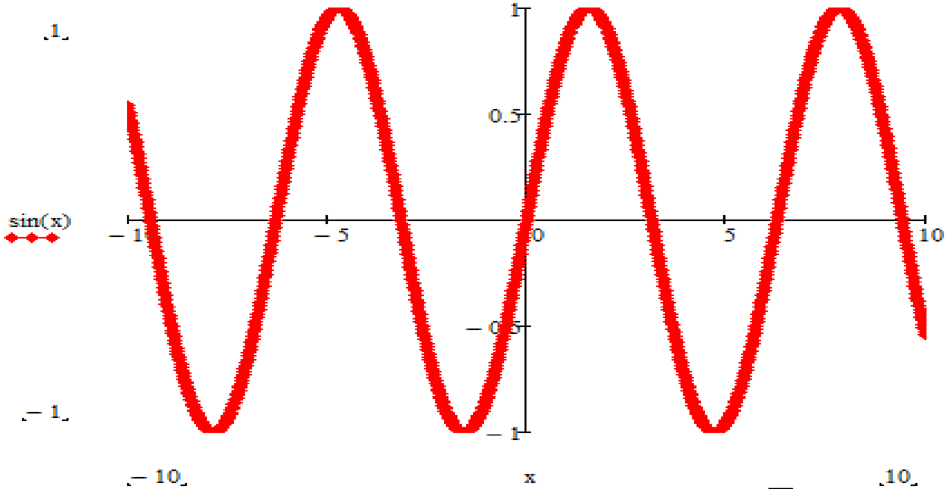
Example: Evaluate the value of $\log_3 2 \cdot \log_4 3 \cdot \log_5 4$.

By alternate form of the change of base formula, $\mathbf{\log_b a \cdot \log_c b = \log_c a}$. We apply this twice to evaluate the given expression.

$$\log_3 2 \cdot \log_4 3 \cdot \log_5 4 = \log_4 2 \cdot \log_5 4 = \log_5 2$$

The Trigonometric Functions

$$1-\ y = f(x) = \sin x \quad \text{and} \quad y = f(x) = \cos x$$

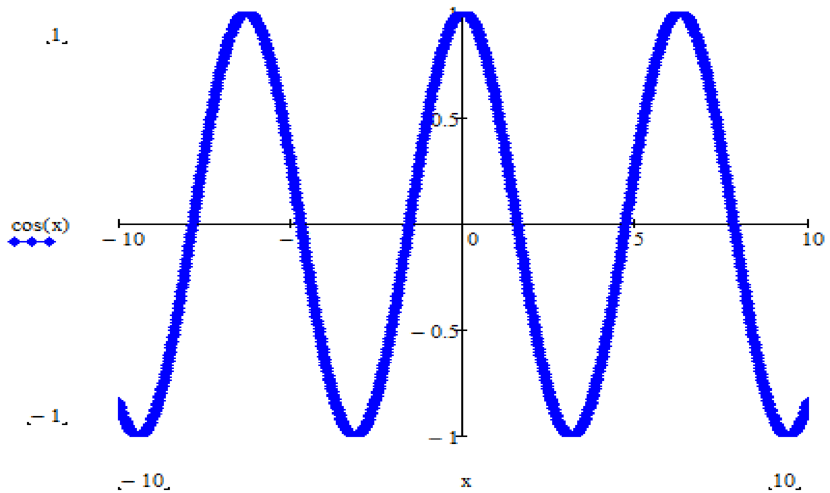


$$D_f = R \quad \text{and} \quad R_f = [-1,1]$$

$$\sin(x+2n\pi) = \sin x \quad \text{and} \quad \sin(-x) = -\sin x$$

The Trigonometric Functions

$$1-\ y = f(x) = \sin x \quad \text{and} \quad y = f(x) = \cos x$$

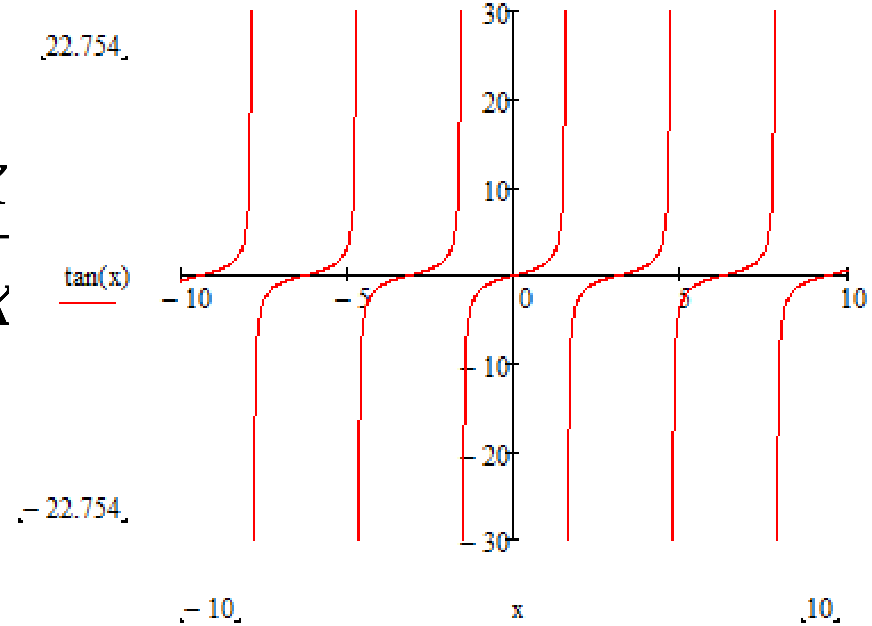


$$D_f = R \quad \text{and} \quad R_f = [-1,1]$$

$$\cos(x + 2n\pi) = \cos x \quad \text{and} \quad \cos(-x) = \cos x$$

The Trigonometric Functions

$$2-y=f(x)=\tan x=\frac{\sin x}{\cos x}$$



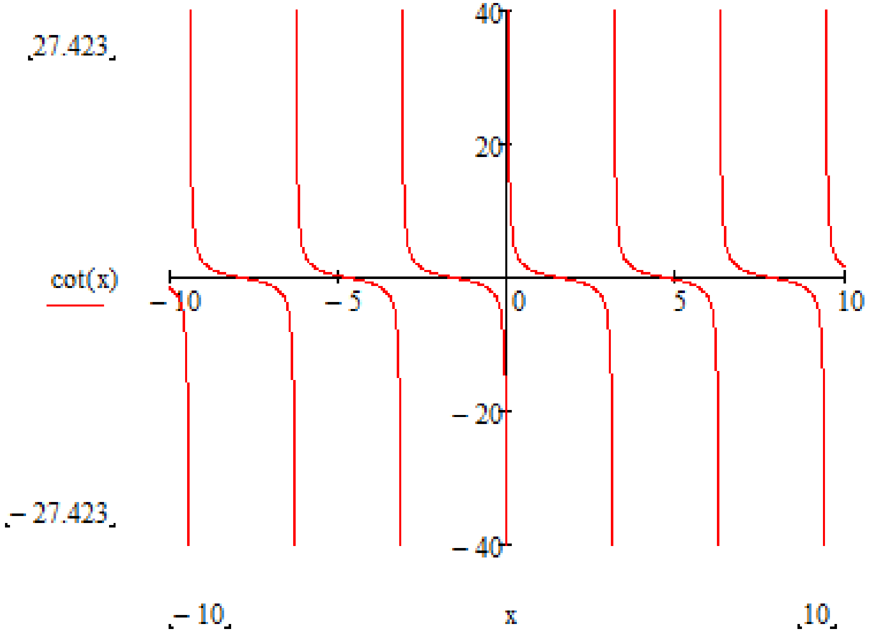
$$D_f=\{x\,|\,x\in R,\cos x\neq 0\}$$

$$=\left\{x\,|\,x\in R,x\neq\frac{n\pi}{2}(n=\mp 1.\mp 3,\mp 5,\cdots)\right\}$$

$$R_f=R$$

The Trigonometric Functions

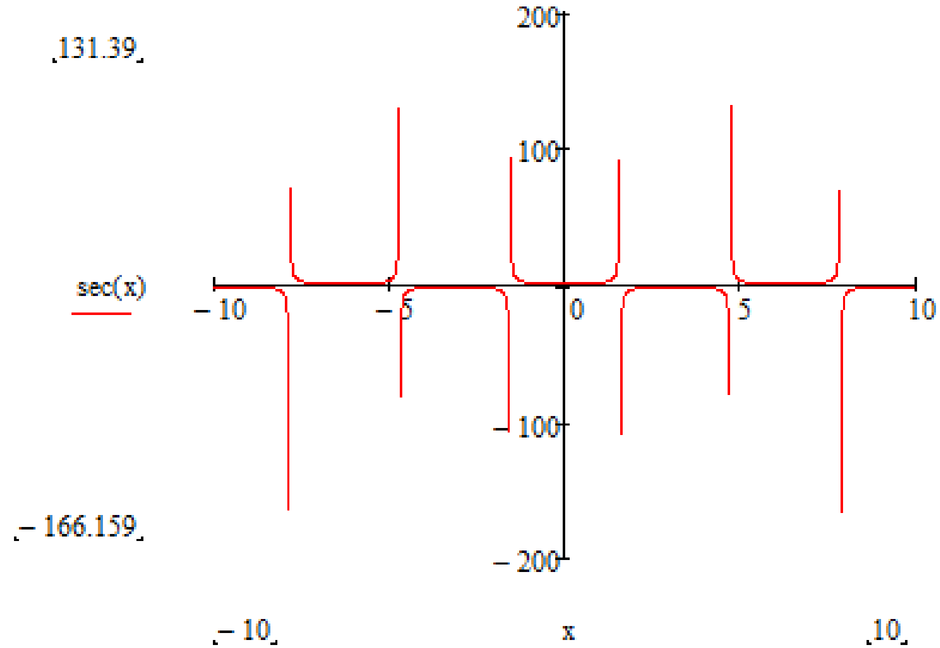
$$3-\ y = f(x) = \cot x = \frac{\cos x}{\sin x}$$



$$D_f = \{x \mid x \in R, \sin x \neq 0\}$$
$$= \{x \mid x \in R, x \neq n\pi (n = 0, \mp 1, \mp 2, \cdots)\}$$
$$R_f = R$$

The Trigonometric Functions

$$4- y = f(x) = \sec x = \frac{1}{\cos x}$$



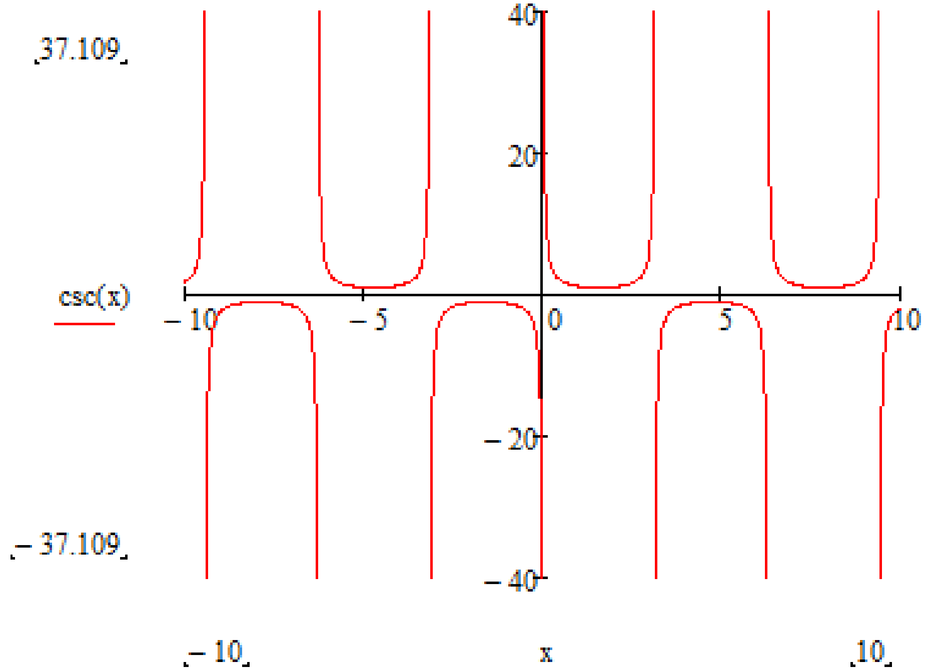
$$D_f = \{x \mid x \in R, \cos x \neq 0\}$$

$$= \left\{x \mid x \in R, x \neq \frac{n\pi}{2} (n = \mp 1, \mp 3, \cdots)\right\}$$

$$R_f = R - (-1,1) \text{ or } |y| \geq 1$$

The Trigonometric Functions

$$5-\ y = f(x) = \csc x = \frac{1}{\sin x}$$



$$D_f = \{x \mid x \in R, \sin x \neq 0\}$$

$$= \{x \mid x \in R, x \neq n\pi (n = 0, \mp 1, \mp 2, \cdots)\}$$

$$R_f = R - (-1,1) \text{ or } |y| \geq 1$$